AFFINE STACKS AND DERIVED RINGS

AKHIL MATHEW AND SHUBHODIP MONDAL

Abstract. In this paper, we use derived rings to revisit the theory of affine stacks due to Toën and introduce a class of derived stacks called affine derived stacks. We discuss several applications and examples, namely, in p-adic homotopy theory, formal groups, and prismatization.

Contents

1. Introduction

Let k be a commutative ring. In the following, a *stack* denotes an fpqc sheaf of spaces on the category of k-algebras. In [Toë06, Def. 2.2.4], Toën defines the subclass of *affine stacks* over k and shows an anti-equivalence (induced by taking global sections) between affine stacks over k and the homotopy theory of the model category of cosimplicial k-algebras, cf. [Toë06, Cor. 2.2.3]. We refer also to [MRT22] and [MR23] for some recent applications of the theory of affine stacks.

The class of affine stacks can be described intrinsically as the smallest class of stacks containing the objects $K(\mathbb{G}_a, n)$ and stable under limits. When k is a field, a pointed connected stack is affine if and only if the homotopy group sheaves are representable by unipotent group schemes (possibly of infinite type) [Toë06, Th. 2.4.1].

When k is a field of characteristic zero, the theory of affine stacks over k is also developed by Lurie, [Lur11b, Sec. 4], at least in the connected case, in which case affine stacks correspond to coconnective E_∞-algebras R/k such that $k \simeq \pi_0(R)$, cf. [Lur11b, Th. 4.4.1]. Moreover, Lurie shows that quasi-coherent sheaves on the affine stack corresponding to R can be identified with the left-completion of $\mathrm{Mod}(R)$, cf. [Lur11b, Prop. 4.5.2]. The work [Lur11b] uses E_∞-rings rather than cosimplicial rings, and therefore requires a restriction to characteristic zero.

The purpose of this paper is to revisit the theory of affine stacks using the theory of derived rings as developed by Raksit [Rak20, Sec. 4] (instead of using the model category of cosimplicial

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rings), and to generalize some of the results of Lurie and Toën to the setting of derived algebraic geometry.

The key idea in this paper is approaching the above story via descent: in Section 2, we introduce the notion of coconnective faithful flatness, which may be viewed as an extension of the notion of faithful flatness for animated rings due to Lurie to the set up of derived rings. In particular, we observe that (derived) affine stacks satisfy descent along coconnectively faithfully flat maps, which allows us to reprove and extend the results discussed above. To elaborate further, we fix a base ring k. Let us first define the class of affine derived k-stacks.

Definition 1.1 (Affine derived stacks)**.** Let Ani_k be the ∞-category of animated k-algebras. A *derived k-stack* is an fpqc hypersheaf of anima[1] on $\mathrm{Ani}_k^{\mathrm{op}}$. A derived k-stack X is said to be *affine* if the following conditions hold.

(1) X is nilcomplete, i.e., for all $A \in \mathrm{Ani}_k$, we have $X(A) \xrightarrow{\sim} \varprojlim X(\tau_{\leq n} A)$.
(2) For each n, the restriction of X to the subcategory of n-truncated animated k-algebras belongs to the subcategory of sheaves of anima on n-truncated animated k-algebras generated under limits by $K(\mathbb{G}_a, i), i \geq 0$.

An affine derived k-stack is said to be *n-derived* for some $n \geq 0$ if, as a functor, it is left Kan extended from the subclass of n-truncated animated k-algebras.

Unlike [Toë06], we allow hypersheaves of anima on all animated rings rather than only on discrete rings. We show that 0-derived affine stacks in the sense of Definition 1.1 agree with affine stacks in the sense of [Toë06].

We write DAlg_k as the ∞-category of derived k-algebras. Let us recall [Rak20, Sec. 4] that DAlg_k provides a way of considering "nonconnective" animated rings. The ∞-category DAlg_k is defined as the ∞-category of algebras over the derived symmetric algebra monad, Sym^*, on $\mathrm{Mod}(k)$.

Construction 1.2 (The derived Spec construction)**.** Given a derived k-algebra $R \in \mathrm{DAlg}_k$, we define a derived stack $\mathrm{Spec}\, R$ which carries an animated k-algebra A to $\mathrm{Hom}_{\mathrm{DAlg}_k}(R, A)$.

It is not difficult to see that for any $R \in \mathrm{DAlg}_k$, $\mathrm{Spec}\, R$ is an affine derived k-stack. The construction $\mathrm{Spec} : \mathrm{DAlg}_k \to \mathrm{Stacks}^{\mathrm{op}}$ is a left adjoint; its right adjoint is given as follows. Given any stack X, the coherent cohomology $R\Gamma(X, \mathcal{O}) \in D(k)$ naturally has the structure of an object of DAlg_k, and $X \mapsto R\Gamma(X, \mathcal{O}_X)$ is the right adjoint. Our first main result states that under mild hypotheses, affine derived stacks canonically arise in this form.

Theorem 1.3. *The functor* $\mathrm{Spec} : \mathrm{DAlg}_k^{\mathrm{op}} \to \mathrm{Stacks}$ *is fully faithful on the subcategory of derived k-algebras which are either bounded-below or bounded-above. The image of* Spec *on the subcategory of n-truncated derived k-algebras is precisely the subcategory of n-derived affine k-stacks.*

Theorem 1.3 generalizes [Toë06, Cor. 2.2.3] (which is precisely the case of 0-derived affine stacks) or [Lur11b, Th. 4.4.1] (in characteristic zero, and with a connectivity hypothesis) to a description of n-derived affine stacks for any n in terms of derived rings. As a result, we obtain (cf. Corollary 3.14) a description of the ∞-category of affine derived k-stacks as the limit of the ∞-categories of n-truncated derived k-algebras. As a corollary of Theorem 1.3, we see that the ∞-category underlying the model category of cosimplicial algebras is canonically equivalent to the ∞-category of coconnective derived rings (see Corollary 3.9). A variant of Corollary 3.9 (without the refinement of coconnective objects) has also been obtained recently by Brantner–Campos–Nuiten using model theoretic methods (see [LBN23, Cor. 5.29]).

Our second main result gives a comparison for quasi-coherent sheaves on an affine stack, generalizing a result for pointed connected affine 0-derived stacks over fields of characteristic zero

[1] Also called the ∞-category of ∞-groupoids or the ∞-category of spaces.

due to Lurie [Lur11b, Prop. 4.5.2] and fields of arbitrary characteristic due to Mondal–Reinecke [MR23, Prop. 2.2.28]).

Theorem 1.4. *Let R be a derived k-algebra. The functor*

$$\mathrm{Mod}(R) \to \mathrm{QCoh}(\operatorname{Spec} R)$$

establishes the target as the left-completion of the source (with respect to the t-structure of Definition 2.1).

Using the coconnectively faithfully flat descent technique, we prove the following result, which generalizes a result due to Toën for stacks over a field [Toë06, Th. 2.4.1].

Theorem 1.5. *Let k be a ring such that every finitely presented k-module has finite flat dimension. Let X be a pointed connected hypercomplete fpqc sheaf on discrete k-algebras. Assume that the homotopy sheaves $\pi_i(X)$ are representable by unipotent affine group schemes (see Definition 4.12) over k. Then X is an affine stack.*

Remark 1.6. Note that every regular noetherian ring k satisfies the assumptions of the above theorem. The converse of Theorem 1.5 does not hold unless k is a field; see [MR23, Ex. 5.4.1].

We also discuss some applications and examples of affine stacks and the corresponding derived rings. In particular, we show that quasi-affine schemes and the classfiying stacks of group schemes that are Cartier duals to formal groups are all affine stacks (see Proposition 4.7 and Proposition 4.19). We also recall a result of Bhatt–Lurie that the stacks arising as the relative prismatization [BL22b, Dri20] of an affine p-adic formal scheme are affine. As a consequence, we recover an unpublished result of Bhatt (Corollary 4.21) which identifies modules over derived prismatic cohomology with quasi-coherent sheaves on the prismatization.

Next, we use the theory of affine stacks to give an exposition of p-adic homotopy theory over $\mathbb{F}_p$-schemes, following ideas of Lurie [Lur13] (who considers E_∞-rings rather than derived rings). In [Lur13, Sec. 2], the notion of a *p-constructible sheaf* of anima on a qcqs scheme X is introduced, defining a subcategory $\mathrm{Shv}_{p\text{-cons}}(X)$ of the ∞-category of étale sheaves of anima on X; in particular, this condition implies that the homotopy groups of each stalk are finite p-groups.

Theorem 1.7. *Let A be an $\mathbb{F}_p$-algebra. Then there is a fully faithful functor*

$$C^*(-,\mathcal{O}) : \mathrm{Pro}(\mathrm{Shv}_{p\text{-cons}}(\operatorname{Spec} A))^{\mathrm{op}} \to \mathrm{DAlg}_A.$$

In the case where A is regular noetherian, the essential image consists of those derived A-algebras whose homotopy groups are filtered colimits of finitely generated unit Frobenius modules in the sense of [EK04].

Finally, we prove the following theorem, which generalizes [MR23, Thm. 1.0.8] over an arbitrary base ring.

Theorem 1.8. *Let k be a discrete ring. Let $\mathcal{G}_k$ denote the ∞-category of augmented coconnective derived rings R over k such that $H^2(R)$ is a projective module of rank r over k, and $H^*(R) \simeq \mathrm{Sym}^* H^2(R)$. Then $\mathcal{G}_k$ is equivalent to the category of commutative formal groups of dimension r.*

The proof of the above result once again relies on the notion of coconnective faithful flatness. Relatedly, in Proposition 6.4, we prove that if F is a formal group over an arbitrary base ring k, then the stacks $K(F^\vee, n)$ are affine. In Remark 6.10, we explain how one can recover and extend the 1-dimensional formal group constructed by [Dri21] to the stack $(\operatorname{Spf}\mathbb{Z}_p)^{\mathrm{syn}}$ by using Theorem 1.8 and the derived ring obtained from (Nygaard filtered) prismatic cohomology of $B\mathbb{G}_m$. This extension has recently been obtained by Manam using topological periodic cyclic homology, who also proves an algebraization result for this formal group [Man24].

Conventions. We will freely use the language of ∞-categories and "higher algebra" as developed in [Lur09], [Lur17]. We will denote by $\mathcal{S}$ the ∞-category of spaces, also called the ∞-category of anima or ∞-groupoids. For an ordinary base ring k, we use Ani_k to denote the category of animated k-algebras and $\mathrm{Ani}_{k,\leq n}$ to denote the category of n-truncated animated k-algebras. We denote by DAlg the ∞-category of derived rings [Rak20]; for $A \in \mathrm{DAlg}$, we use DAlg_A to denote $\mathrm{DAlg}_{A/}$. The category of n-truncated derived A-algebras are denoted by $\mathrm{DAlg}_{A,\leq n}$. An fpqc hypersheaf of anima on $\mathrm{Ani}_k^{\mathrm{op}}$ will be called a derived k-stack.

For an $\mathbb{F}_p$-algebra R, we let $R_{\mathrm{perf}} := \varinjlim_{\varphi} R$ to be the direct limit perfection of R.

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2. Coconnective faithful flatness

In this section, we discuss a notion of faithful flatness for maps of derived rings (which reduces to the usual notion in the connective case, i.e., for animated rings) and prove an analog of faithfully flat descent.

Definition 2.1. Let R be any E_∞-ring. Using [Lur18, Prop. C.6.3.1], we can define a t-structure on $\mathrm{Mod}(R)$ as follows:

(1) The connective objects $\mathrm{Mod}(R)_{\geq 0}$ are the smallest subcategory of $\mathrm{Mod}(R)$ generated under colimits and extensions by R itself.
(2) The coconnective objects $\mathrm{Mod}(R)_{\leq 0}$ are the objects of $\mathrm{Mod}(R)$ which are coconnective as underlying spectra.

By *loc. cit.*, the t-structure on $\mathrm{Mod}(R)$ is compatible with filtered colimits.

Remark 2.2. When R is connective, the above is the usual t-structure on $\mathrm{Mod}(R)$, i.e., an object of $\mathrm{Mod}(R)$ is (co)connective if and only if the underlying spectrum is (co)connective.

Let $A \to A'$ be a map of derived rings. It is easy to see that the base-change functor $- \otimes_A A' : \mathrm{Mod}(A) \to \mathrm{Mod}(A')$ is right t-exact: that is, it preserves connective objects.

Definition 2.3 (Coconnective flatness)**.** We say that $A \to A'$ is *coconnectively flat* if extension of scalars $\mathrm{Mod}(A) \to \mathrm{Mod}(A')$ is left t-exact (and therefore t-exact): in other words, if extension of scalars preserves coconnective objects.

Definition 2.4 (Coconnectively faithful flatness)**.** Let $A \to A'$ be a map of derived rings. We say that it is *coconnectively faithfully flat* if it is coconnectively flat and extension of scalars is conservative on bounded-above objects: that is, an A-module which is n-truncated for some n vanishes if and only if its extension of scalars to A' vanishes.

Proposition 2.5. *The map $A \to A'$ is coconnectively faithfully flat if and only if for every $M \in \mathrm{Mod}(A)_{\leq 0}$, the map*

$$M \to M \otimes_A A'$$

has fiber in $\mathrm{Mod}(A)_{\leq -1}$.

Proof. Suppose $M \otimes_A \mathrm{fib}(A \to A')$ belongs to $\mathrm{Mod}(A)_{\leq -1}$ for all $M \in \mathrm{Mod}(A)_{\leq 0}$, so it follows that $M \otimes_A A' \in \mathrm{Mod}(A)_{\leq 0}$. Thus, $A \to A'$ is coconnectively flat. Moreover, the hypothesis implies that the map $M \to M \otimes_A A'$ induces an injection on homotopy groups (with respect to the t-structure); this implies that if M is bounded-below, then M vanishes if and only if $M \otimes_A A'$ vanishes.

Conversely, suppose that $A \to A'$ is coconnectively faithfully flat. Given $M \in \mathrm{Mod}(A)_{\leq 0}$, we wish to show that $\mathrm{fib}(M \to M \otimes_A A') \in \mathrm{Mod}(A)_{\leq 0}$. In fact, it suffices to show that $M \to M \otimes_A A'$ induces an injection on homotopy groups (with respect to the t-structure); however, this can be tested after extension of scalars to A', where the map has a section. $\square$

Proposition 2.6. *The class of coconnectively faithfully flat maps of derived rings is closed under composition, base change, and filtered colimits.*

Proof. The closure under composition and base change follow directly from Definition 2.4. The closure under filtered colimits follows from Proposition 2.5. □

Construction 2.7 (A Grothendieck topology on DAlg)**.** As a consequence of Proposition 2.6, by [Lur18, Prop. A.3.2.1], there is a Grothendieck topology on $\mathrm{DAlg}^{\mathrm{op}}$ with the covers generated by the coconnectively faithfully flat maps and the maps $A_1 \times A_2 \to A_1, A_1 \times A_2 \to A_2$ for $A_1, A_2 \in \mathrm{DAlg}$.

Remark 2.8. By iteratively using Definition 2.4, it follows that if $A \to A'$ is a coconnectively faithfully flat map of derived rings and $M \in \mathrm{Mod}(A)_{\leq 0}$ is such that $M \otimes_A A' \simeq 0$, then $M \simeq 0$.

Remark 2.9 (Comparison with the connective case)**.** Let $A \to A'$ be a map of *animated* rings. Then the map $A \to A'$ is coconnectively faithfully flat if and only if it is faithfully flat in the sense of [Lur11a, Def. 5.2]: that is, if $\pi_0(A) \to \pi_0(A')$ is faithfully flat as a map of discrete rings, and $\pi_*(A') \simeq \pi_0(A') \otimes_{\pi_0(A)} \pi_*(A)$. This follows from [Lur17, Th. 7.2.2.15].

Remark 2.10. Let A be a derived ring. Let A_0 be an animated ring with a map $A_0 \to A$, and let $A_0 \to B_0$ be a map of animated rings. Let $B = A \otimes_{A_0} B_0$, so that $\mathrm{Mod}(B)$ can be described as B_0-modules in the (A_0-linear) ∞-category $\mathrm{Mod}(A)$. The forgetful functor $\mathrm{Mod}(B) \to \mathrm{Mod}(A)$ is t-exact (left t-exactness is automatic, and right t-exactness follows because B is connective as A-module). Thus, $\mathrm{Mod}(B)^\heartsuit$ can be described as the abelian category of $\pi_0(B_0)$-modules in the $\pi_0(A_0)$-linear abelian category $\mathrm{Mod}(A)^\heartsuit$.

We will frequently use the following criterion for coconnective faithful flatness; when A is a field, the result is [Lur11b, Cor. 4.1.12].

Proposition 2.11. *Let A be an animated ring. Let R be an augmented derived A-algebra, and suppose the fiber of the augmentation $f : R \to A$ has* Tor*-amplitude in degrees ≤ -1 as A-module. Then the augmentation $f : R \to A$ is coconnectively faithfully flat.*

Lemma 2.12 (Small object argument)**.** *Let $\mathcal{C}$ be a presentable stable ∞-category, and let $\mathcal{C}_0 \subset \mathcal{C}$ be a subcategory consisting of compact objects. Given any object $X \in \mathcal{C}$, there exists a map $X \to X'$ in $\mathcal{C}$ such that:*

(1) X' receives no nonzero maps from any object in $\mathcal{C}_0$.
(2) The fiber of $X \to X'$ belongs to the smallest subcategory of $\mathcal{C}$ generated under extensions and sequential colimits under $\mathcal{C}_0$.

Proof. Follows from iteratively taking cofibers of maps from objects of $\mathcal{C}_0$, cf. [Lur18, Prop. 12.4.2.1]. □

Lemma 2.13. *Let A be an animated ring. Let R be an augmented derived A-algebra, and suppose the fiber of the augmentation $f : R \to A$ has* Tor*-amplitude in degrees ≤ -1 as A-module. Consider the following properties for an object $N \in \mathrm{Mod}(R)$:*

(1) The underlying A-module of N has Tor*-amplitude in degrees ≤ 0, and*
(2) For any $N' \in \mathrm{Mod}(R)_{\leq 0}$, we have $N \otimes_R N' \in \mathrm{Mod}(R)_{\leq 0}$ (so in some sense, N has Tor*-amplitude in degrees ≤ 0 as an R-module).*

Then property (1) implies property (2).

Proof. To this end, let $\mathcal{C}_1 \subset \mathrm{Mod}(R)$ denote the subcategory of objects satisfying (1), and $\mathcal{C}_2 \subset \mathcal{C}_1$ the subcategory of objects satisfying both (1) and (2). We need to show that the inclusion $\mathcal{C}_2 \subset \mathcal{C}_1$ is actually an equivalence.

Note that both $\mathcal{C}_1$ and $\mathcal{C}_2$ are closed under filtered colimits, finite limits, and extensions. By our assumption, it follows that R belongs to $\mathcal{C}_2$.

Given $N \in \mathcal{C}_1$, we start by finding a map $N \to N_1$ such that:

(a) $N_1 \in \mathcal{C}_1$
(b) $\mathrm{fib}(N \to N_1) \in \mathcal{C}_2$
(c) $N \to N_1$ induces the zero map on π_*.

To construct N_1 given N, we start by applying the small object argument (Lemma 2.12) to the class $\mathcal{C}_0$ spanned by $R[-i], i \geq 1$. This produces a map $N \to N'$ such that $\pi_i(N') = 0$ for $i < 0$, $\mathrm{fib}(N \to N') \in \mathcal{C}_2$ (as an iterative extensions of $R[-i], i > 0$), and such that $N' \in \mathcal{C}_1$.

Since N' has Tor-amplitude in degrees ≤ 0 as A-module, it follows that N' (which is connective) is flat as an A-module. We now set $N_1 = \mathrm{cofib}(N' \otimes_A R \to N')$, and the map $N \to N_1$ is defined to be the composition $N \to N' \to N_1$. One sees that the desired conditions are satisfied. Indeed, (a) follows from the fact that $f : R \to A$ has Tor-amplitude in degrees ≤ -1 as A-module. Note that $N' \otimes_A R \in \mathcal{C}_2$, because N' can be expressed as a filtered colimit of finite, free A-modules, and $\mathcal{C}_2$ is closed under colimits. Thus, the condition (b) follows from the facts that $\mathrm{fib}(N \to N') \in \mathcal{C}_2$, and $\mathcal{C}_2$ is closed under extensions. Condition (c) follows directly from the construction.

Now, given the process $N \mapsto (N \to N_1)$, we repeat a countable number of times to obtain a sequence $N \to N_1 \to N_2 \to \dots$ which induce the zero maps on π_* and whose successive fibers belong to $\mathcal{C}_2$. Passing to the colimit, we find that $N \in \mathcal{C}_2$ as desired. □

Proof of Proposition 2.11. Applying Lemma 2.13 to $\mathrm{fib}(R \to A)[1]$ gives the result. □

Corollary 2.14. *The map* $\mathrm{Sym}^*_{\mathbb{Z}}(\mathbb{Z}[-n]) \to \mathbb{Z}$ *is coconnectively faithfully flat for any* $n > 0$*.*

Proof. This follows from Proposition 2.11. In fact, the needed Tor-amplitude assumption follows by base-change to any residue field; if k is a field and $V \in \mathrm{Mod}(k)_{\leq -1}$, then $\mathrm{Sym}^i V \in \mathrm{Mod}(k)_{\leq -1}$ for all $i > 0$. □

Remark 2.15. The analog of this result does not appear to hold in spectral algebraic geometry over $\mathbb{Z}$ (i.e., with E_∞-algebras over $\mathbb{Z}$ rather than derived rings over $\mathbb{Z}$). We do not know if there is an analog of the theory of affine stacks in spectral algebraic geometry.

Proposition 2.16. *Given any derived ring A, there exists a map $A \to A'$ which is coconnectively faithfully flat and such that A' is an animated ring.*

Proof. This follows by transfinitely iterating Corollary 2.14. □

We now prove an analog of faithfully flat descent, following ideas of [Lur18, Sec. D.6]. To this end, we make the construction $A \to \mathrm{Mod}(A)_{\leq 0}$ into a functor of $A \in \mathrm{DAlg}$ (with values in presentable ∞-categories), via extension of scalars followed by truncation in the t-structure.

Proposition 2.17 (Flat hyperdescent)**.** *The construction* $A \mapsto \mathrm{Mod}(A)_{\leq 0}$ *satisfies hyperdescent in coconnectively faithfully flat topology on* DAlg.

Proof. First, we show that $A \mapsto \mathrm{Mod}(A)_{\leq 0}$ satisfies descent for the coconnectively faithfully flat topology. Since the construction preserves finite products, it suffices [Lur18, Prop. A.3.3.1] to show that if $A \to B$ is a map in DAlg which is coconnectively faithfully flat and $B^\bullet_+$ is the augmented Čech nerve (so $B_{-1} = A$), then $\mathrm{Mod}(B^\bullet_+)$ is a limit diagram in Cat_∞.

We verify this using the ∞-categorical monadicity theorem, in the form of [Lur17, Cor. 4.7.5.3]. The adjointability condition is automatic in light of [Lur18, Lem. D.3.5.6] (while we are working with coconnective objects, all base-changes involved in the adjointability condition are along coconnectively faithfully flat morphisms), so it suffices to show the functor $\mathrm{Mod}(A)_{\leq 0} \to \mathrm{Mod}(B)_{\leq 0}$ is comonadic (cf. [Lur18, Lem. D.3.5.7] for this argument as well), and it suffices to show that the extension of scalars functors $\mathrm{Mod}(A)_{\leq 0} \to \mathrm{Mod}(B)_{\leq 0}$ preserves totalizations and is conservative. Both follow because the extension of scalars functor is t-exact and conservative on the heart, by the assumption of coconnective faithful flatness. This proves the comonadicity and we deduce that $A \mapsto \mathrm{Mod}(A)_{\leq 0}$ is a sheaf or the coconnectively faithfully flat topology.

Now we prove hyperdescent. Let $\Delta_{s,+}$ be the augmented semisimplicial category. Thanks to [Lur18, Prop. A.5.7.2], it suffices to show that if $C^\bullet : \Delta_{s,+} \to \mathrm{DAlg}$ is a hypercovering for the coconnectively faithfully flat topology in the sense of [Lur18, Def. A.5.7.1], then $\mathrm{Mod}(C^\bullet)_{\leq 0}$ is a limit diagram in Cat_∞. But we have (as augmented cosemisimplicial objects) $\mathrm{Mod}(C^\bullet)_{\leq 0} \simeq \lim_n \mathrm{Mod}(C^\bullet)_{[-n,0]}$ under the truncation functors. Now $\mathrm{Mod}(C^\bullet)_{[-n,0]}$ is a limit diagram by [Lur18, Prop. A.5.7.2] again because $A \mapsto \mathrm{Mod}(A)_{[-n,0]}$ is (by what was proved above) a sheaf for the coconnectively faithfully flat topology and is necessarily hypercomplete because it takes values in $(n+1)$-categories. □

Corollary 2.18. *For any n and $A \in \mathrm{DAlg}$, let $\mathrm{DAlg}_{A,\leq n}$ denote the category of n-truncated derived rings. Then the construction $A \mapsto \mathrm{DAlg}_{A,\leq n}$ satisfies hyperdescent in the coconnectively faithfully flat topology on* DAlg*.*

Proof. This is proved in a way similar to Proposition 2.17. □

Construction 2.19. Given $A \in \mathrm{DAlg}_k$, we let $\tau_{\leq n}^{\mathrm{Mod}(A)} A$ denote the n-truncation of A in the ∞-category $\mathrm{Mod}(A)$ with its t-structure constructed above.

Remark 2.20. Suppose $n \geq 0$. We observe that $\tau_{\leq n}^{\mathrm{Mod}(A)} A$ has the structure of an object in DAlg_k, and is the universal n-truncated derived k-algebra that A maps to. To see this, we observe that the universal n-truncated derived k-algebra that A maps to is obtained from A by repeatedly forming pushouts along the map $\mathrm{Sym}^*(k[i]) \to k$ for $i > n$, and this does not change the $\mathrm{Mod}(A)$-homotopy groups in degrees $\leq n$. To emphasize this, we will sometimes write $\widetilde{\tau_{\leq n}}$ for this truncation. Note that the category $\mathrm{DAlg}_{A,\leq n}$ has all colimits.

Corollary 2.21. *On the ∞-category* DAlg*, the construction $A \mapsto \varprojlim_n \tau_{\leq n}^{\mathrm{Mod}(A)} A$ satisfies hyperdescent (as a functor to spectra) for the coconnective faithfully flat topology.*

Proof. This follows from Proposition 2.17 upon taking the limit in n. □

Question. Let R be a derived ring which is bounded-below, i.e., $\pi_i R = 0$ for $i < -d$. Then it is easy to see that the t-structure on $\mathrm{Mod}(R)$ is hypercomplete, i.e., there are no ∞-connective objects (because any n-connective object has $(n-d)$-connective underlying spectrum). Under what conditions is the t-structure Postnikov complete, i.e., when does $\mathrm{Mod}(R) \simeq \varprojlim_n \mathrm{Mod}(R)_{\leq n}$?

While we do not know the answer to the above question in general, we include the following criterion for left-completeness.

Proposition 2.22. *Let R be a derived ring. Suppose there is a map $R \to R'$ such that:*

(1) The map is descendable (in the sense of [Mat16, Def. 3.18]*, or* [Lur18, Sec. D.3]*).*

(2) R' and $R' \otimes_R R'$ are animated rings.

Then the t-structure on $\mathrm{Mod}(R)$ is left-complete.

Proof. Let $R'^\bullet$ denote the cosemisimplicial Čech nerve of $R \to R'$; then we have $\mathrm{Mod}(R) \simeq \varprojlim \mathrm{Mod}(R'^\bullet)$ by [Mat16, Prop. 3.22] (and [Lur09, Lem. 6.5.3.7] to replace cosimplicial with cosemisimplicial objects). Since we have used the semisimplicial Čech nerve, all the functors involved are t-exact. By assumption, all the terms occurring in the inverse limit are module ∞-categories over animated rings and hence left-complete. The result follows. □

3. Embedding into derived stacks

Let us fix a base ring k. In this section we discuss the functor ("Spec") from derived k-algebras to derived k-stacks. The main results are that this functor is fully faithful on derived rings which are homologically either bounded-below or bounded-below (Theorem 3.6), and that one has a comparison between modules and quasi-coherent sheaves (Theorem 3.18).

We recall the following general fact. Let $\mathcal{T}$ be a site, and let $\mathcal{B} \subset \mathcal{T}$ be a basis. Then hypercomplete sheaves of anima on $\mathcal{T}$ are equivalent to hypercomplete sheaves of anima on $\mathcal{B}$, via right Kan extension, [Aok, Th. A.6]. Thus, we obtain:

Corollary 3.1. *Let $\mathcal{C}$ be an ∞-category with all limits. Then for a functor $f : \mathrm{DAlg} \to \mathcal{C}$, the following are equivalent:*

(1) f is a hypercomplete sheaf for the coconnectively faithfully flat topology.
(2) f is right Kan extended from animated rings, and restricts to a hypercomplete sheaf for the flat topology on animated rings.

Definition 3.2. Let Ani(Ring) denote the ∞-category of animated rings. We equip Ani(Ring) (or rather its opposite) with the faithfully flat topology. Let Stacks be the ∞-category of hypersheaves of anima on Ani(Ring).

Construction 3.3 (The spectrum)**.** Given a derived ring $R \in \mathrm{DAlg}$, we obtain a stack $\mathrm{Spec}\, R \in$ Stacks carrying an animated k-algebra A to $\mathrm{Hom}_{\mathrm{DAlg}_k}(R, A)$.

Proposition 3.4. *The construction* $\mathrm{Spec} : \mathrm{DAlg}_k \to \mathrm{Stacks}^{\mathrm{op}}$ *preserves colimits; its right adjoint carries a stack X to the coherent cohomology $R\Gamma(X, \mathcal{O})$ (considered as a derived ring).*

Proof. The first claim is evident. The second claim amounts to the assertion that if Y is any stack and $R \in \mathrm{DAlg}_k$, then $\mathrm{Hom}_{\mathrm{Stacks}}(Y, \mathrm{Spec}\, R) = \mathrm{Hom}_{\mathrm{DAlg}_k}(R, R\Gamma(Y, \mathcal{O}_Y))$; this reduces by taking colimits to the case where Y is affine, where it follows from the definitions. □

Proposition 3.5. *The construction* $\mathrm{Spec} : \mathrm{DAlg}_k \to \mathrm{Stacks}^{\mathrm{op}}$ *carries coconnectively faithfully flat morphisms into surjections of fpqc sheaves.*

Proof. Let $R \to R'$ be a coconnectively faithfully flat morphism in DAlg_k. Given a map $R \to A$ with A an animated ring, we need to find a faithfully flat animated A-algebra B such that $R \to A \to B$ extends over R'. To this end, we form $R' \otimes_R A$, which is a derived A-algebra which is coconnectively faithfully flat over A. By Proposition 2.16, we can find an animated ring B over $R' \otimes_R A$ which is coconnectively faithfully flat; the composite $A \to R' \otimes_R A \to B$ is coconnectively faithfully flat, whence faithfully flat. □

We now prove Theorem 1.3, which we restate for convenience.

Theorem 3.6 (Cf. [Toë06, Prop. 2.2.2])**.** *The collection of derived rings which either are n-truncated or $(-n)$-connective for some n embeds fully faithfully into the ∞-category of stacks* Stacks *via the functor* Spec*. In particular, for any $R \in \mathrm{DAlg}_k$ which is either n-truncated or $(-n)$-connective for some n, the natural adjunction map*

$$R \to R\Gamma(\mathrm{Spec}\, R, \mathcal{O}) \tag{3.0.1}$$

is an equivalence.

Proof. We claim that the natural map (3.0.1) exhibits the target as the hypercompletion of the source, as presheaves of spectra on DAlg_k. To this end, we observe that the map is known to be an equivalence for R an animated ring (which forms a basis for the topology), while the target is a hypersheaf for the coconnectively faithfully flat topology thanks to Proposition 3.5 and usual faithfully flat descent. The claim now follows from Corollary 3.1.

Next, we claim that the natural map (3.0.1) is isomorphic to the natural map $R \to \varprojlim_m \tau_{\leq m}^{\mathrm{Mod}(R)} R$ of Construction 2.19. In fact, in light of Corollary 2.21, this follows (via Corollary 3.1) because the natural map is an equivalence for animated rings, and the target satisfies hyperdescent for the coconnective faithfully flat topology.

Therefore, we need to show that if R is either n-truncated or $(-n)$-connective, then the map from R to $\varprojlim_m \tau_{\leq m}^{\mathrm{Mod}(R)} R$ is an equivalence. The former claim is immediate. For the latter, we use the description of $\tau_{\leq m}^{\mathrm{Mod}(R)}R$ for $m > 0$ given by Remark 2.20; for $m \gg 0$, we find that $\tau_{\leq m}^{\mathrm{Mod}(R)}R$ is an isomorphism in degrees $< m - n$ (as we are forming base-changes along $\mathrm{Sym}^*_k(k[i]) \to k$ for $i > m$), and the passage to the limit then proves the claim. $\square$

We now show that n-truncated derived k-algebras embed into sheaves of anima on n-truncated animated k-algebras (rather than all animated k-algebras), cf. Corollary 3.8. This could be proved directly using similar arguments, but it will also be helpful to include a complementary observation. The next result is an analog of [Lur11b, Prop. 4.4.4].

Proposition 3.7. *Let $R \in \mathrm{DAlg}_k$. Then there exists an augmented cosimplicial object $R^\bullet \in \mathrm{Fun}(\Delta^+, \mathrm{DAlg}_k)$ such that:*

1. *$R^{-1} = R$.*
2. *Each $R^i, i \geq 0$ is an animated k-algebra.*
3. *The diagram is a hypercover of R^{-1} in the coconnectively faithfully flat topology on DAlg.*
4. *For any animated k-algebra A, the augmented simplicial space $\mathrm{Hom}_{\mathrm{DAlg}_k}(R^\bullet, A)$ is a hypercover.*

Proof. We define a variant of the coconnectively flat topology on DAlg_k as follows. Namely, say that a map $A \to B \in \mathrm{DAlg}_k$ is *coconnectively projective* if it is coconnectively faithfully flat and any map $A \to R$ for R an animated k-algebra extends to B.

Any object of DAlg_k is covered in the coconnectively projective topology by an animated ring; this follows similarly by repeatedly making base-changes along $\bigotimes \mathrm{Sym}^*(k[-n]) \to k$ for $n > 0$ (we only have to do this a countable number of times, so no $\varprojlim^1$-issues arise). $\square$

Corollary 3.8. *The class of n-truncated derived k-algebras embeds contravariantly (via* Spec*) fully faithfully into* $\mathrm{Shv}(\mathrm{Ani}_{k,\leq n})$ *and has image the subcategory of the latter generated under limits by $K(\mathbb{G}_a, i), i \geq 0$. This embedding carries colimits (computed in the ∞-category of truncated k-algebras) to limits in* $\mathrm{Shv}(\mathrm{Ani}_{k,\leq n})$.

Proof. Let R be an n-truncated derived k-algebra. Then by Proposition 3.7, the functor $\mathrm{Spec}\, R$ is left Kan extended from n-truncated animated k-algebras, whence the result follows from Theorem 3.6. $\square$

Corollary 3.9 (Comparison between derived and cosimplicial rings)**.** *Corollary 3.8 (in the case $n = 0$) together with* [Toë06, Cor. 2.2.3] *implies that the underlying ∞-category of the model category of cosimplicial rings and the ∞-category of coconnective derived k-algebras are canonically equivalent.*

Definition 3.10 (Affine derived stacks)**.** A derived k-stack X is said to be *affine* if

1. X is nilcomplete: that is, for any $A \in \mathrm{Ani}_k$, we have $X(A) \xrightarrow{\sim} \varprojlim X(\tau_{\leq m}A)$.
2. For each m, its restriction to $\mathrm{Ani}_{k,\leq m}$ belongs to the subcategory of Corollary 3.8 (equivalently, is corepresentable by an m-truncated derived ring).

Remark 3.11. The subcategory of Stacks given by the affine derived k-stacks is closed under limits.

Example 3.12. For any derived k-algebra R, the functor $\mathrm{Hom}_{\mathrm{DAlg}_k}(R, -)$ is an affine derived k-stack. In fact, its restriction to the subcategory of n-truncated animated k-algebras is given by $\mathrm{Hom}_{\mathrm{DAlg}_k}(\widetilde{\tau_{\leq n}}R, -)$.

Example 3.13. The derived stack $K(\mathbb{G}_a, n)$ is affine: it is the spectrum of the free derived k-algebra, $\mathrm{Sym}^*_k(k[-n])$.

Corollary 3.14. *There is an anti-equivalence of ∞-categories between the ∞-category of affine derived k-stacks and the homotopy limit $\varprojlim_n \mathrm{DAlg}_{k,\leq n}$ where the transition maps are given by $\widetilde{\tau_{\leq n}}$ (using the notation of Remark 2.20).*

Proof. This follows from Corollary 3.8. □

Remark 3.15. Due to the convergence issues, the theory of affine derived stacks behaves slightly differently than the theory of affine stacks in [Toë06].

(1) In general, we do not know how the inverse limit in Corollary 3.14 compares with the ∞-category DAlg_k. When k has characteristic zero, the existence of periodic derived k-algebras (such as $k[t^{\pm 1}]$ with $|t| = 2$) implies that the functor $\mathrm{Spec} : \mathrm{DAlg}_k \to \mathrm{Stacks}_k$ is not fully faithful in general.
(2) Note that by Remark 3.11 and the adjoint functor theorem, for every derived k-stack X, there is a universal map to an affine derived k-stack $X \to \mathrm{U}(X)$. However, there is also a canonical map to an affine derived k-stack, namely, $X \to \mathrm{Spec}\, R\Gamma(X, \mathcal{O}_X)$, which in general is different from the above.

Both these issues do not appear if one restricts attention to n-truncated derived k-algebras throughout (e.g., 0-derived stacks or higher stacks as in [Toë06, Lur11b]), and we expect these issues to be irrelevant in most situations of practical interest.

By abuse of notation, we will refer to a functor on n-truncated animated k-algebras as an *affine derived stack* if it is corepresentable by a derived k-algebra.

In the remainder of the section, we prove the comparison for quasi-coherent sheaves (as in Theorem 1.4, restated for convenience as Theorem 3.18).

Definition 3.16 (Quasi-coherent sheaves)**.** Given a derived k-stack X, we define $\mathrm{QCoh}(X)$ via right Kan extension of $A \mapsto \mathrm{Mod}(A)$ for animated k-algebras A, cf. [Lur18, Def. 6.2.2.1], so

$$\mathrm{QCoh}(X) = \varprojlim_{\mathrm{Spec}\, A \to X, A \text{ animated}} \mathrm{Mod}(A).$$

When $X = \mathrm{Spec}\, R$ for a derived k-algebra R, we have a comparison functor $\mathrm{Mod}(R) \to \mathrm{QCoh}(\mathrm{Spec}\, R)$.

Remark 3.17 (The t-structure on $\mathrm{QCoh}(\mathrm{Spec}\, R)$)**.** If $X = \mathrm{Spec}\, R$ for R a derived k-algebra and we can form a cosimplicial object $R^\bullet$ as in Proposition 3.7, then by Proposition 3.5 and the theory of fpqc descent gives $\mathrm{QCoh}(X) = \varprojlim \mathrm{Mod}(R^\bullet)$. Consequently, we obtain a t-structure on $\mathrm{QCoh}(X)$ such that an object is connective (resp. coconnective) if and only if its pullback to R^0 is connective (resp. coconnective). Note that the t-structure is independent of any choices: an object is connective if and only if its pullback to any affine scheme is connective. However, the explicit identification of the coconnective objects uses the presentation.

The next result is proved in [Lur11b, Prop. 4.5.2] in the case where R is a coconnective k-algebra with $k \simeq H^0(R)$ for k a field of characteristic zero.

Theorem 3.18. *Let $R \in \mathrm{DAlg}_k$. Then the functor $\mathrm{Mod}(R) \to \mathrm{QCoh}(\mathrm{Spec}\, R)$ is t-exact (with respect to the t-structure on the former of Definition 2.1 and on the latter of Remark 3.17), and exhibits the target as the left-completion of the source.*

Proof. The t-exactness follows from the explicit description in Remark 3.17: choose a coconnectively faithfully flat map $R \to R^0$ with R^0 an animated k-algebra, and observe that (by construction) $\mathrm{Mod}(R) \to \mathrm{Mod}(R^0)$ is t-exact. The target $\mathrm{QCoh}(\mathrm{Spec}\, R)$ is left-complete by the description $\mathrm{QCoh}(\mathrm{Spec}\, R) = \varprojlim \mathrm{Mod}(R^\bullet)$ (using the notation of Proposition 3.7). To see the last claim, it thus suffices to verify the equivalence $\mathrm{Mod}(R)_{\leq 0} \xrightarrow{\sim} \varprojlim \mathrm{Mod}(R^\bullet)_{\leq 0}$, which follows from Proposition 2.17. □

Let us point out that the analogue of Theorem 3.18 often holds for stacks that are not affine. In [MR23, Prop. 2.2.28], such a result was proven for pointed connected *weakly* affine stacks ([MR23, Def. 2.2.1]) over a field. In the present paper, we do not discuss the most general statement along these lines, but we note the following case in Proposition 3.21 which would be useful for later applications.

Remark 3.19. In the next result (Proposition 3.21), we work with rings A such that every finitely presented A-module has finite flat dimension. This applies to regular noetherian rings, but also to some non-noetherian examples such as valuation rings or the polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$ in infinitely many variables.

Remark 3.20. The assumption on A appearing above implies that the collection of objects $X \in \mathrm{Mod}(A)$ with Tor-amplitude in degrees ≤ 0 is closed under totalizations. This follows because by a filtered colimit argument, the condition of Tor-amplitude can be tested with respect to finitely presented discrete k-modules. In the latter case the desired condition on Tor-amplitude follows from commutation of totalization; i.e., totalizations of coconnective objects commute with tensors with objects of bounded-above Tor-dimension, as one sees by approximation by finite totalizations.

Proposition 3.21. *Let A be a discrete ring such that every finitely presented A-module has finite flat dimension. Let G be a flat affine commutative group scheme over* $\operatorname{Spec} A$ *and let $n \geq 2$ be an integer. Then the natural functor* $\mathrm{Mod}(R\Gamma(B^nG, \mathcal{O})) \to \mathrm{QCoh}(B^nG)$ *is t-exact and exhibits the target as a left completion of the source.*

Proof. Note that $\mathrm{QCoh}(B^nG)$ is equipped with a t-structure such that an object is connective (resp. coconnective) if and only if its pullback along the natural map $\operatorname{Spec} A \to B^nG$ is connective (resp. coconnective). The t-exactness of $\mathrm{Mod}(R\Gamma(B^nG, \mathcal{O})) \to \mathrm{QCoh}(B^nG)$ follows by construction and because the map $R\Gamma(B^nG, \mathcal{O}) \to A$ is coconnectively faithfully flat; the latter assertion follows from the hypothesis on the ring A and flatness of $G \to \operatorname{Spec} A$ (see Proposition 2.11 and Remark 3.20). By restriction, we obtain a functor

$$U^* : \mathrm{Mod}(R\Gamma(B^nG, \mathcal{O}))_{\leq 0} \to \mathrm{QCoh}(B^nG)_{\leq 0},$$

which we will show to be an equivalence, which would prove the proposition. Note that U^* admits a right adjoint which we denote by $U_* : \mathrm{QCoh}(B^nG)_{\leq 0} \to \mathrm{Mod}(R\Gamma(B^nG, \mathcal{O}))_{\leq 0}$; this can be identified as the global section functor. By choosing an explicit flat hypercover (cf. [MR23, Ex. 2.2.2]) for B^nG, one sees that[2] U_* commutes with filtered colimits. By Remark 2.8 and the fact that $R\Gamma(B^nG, \mathcal{O}) \to A$ is coconnectively faithfully flat, it follows that U^* is conservative. Therefore, to show that U^* is an equivalence, it suffices to show that the counit $U^*U_* \to \mathrm{id}$ is an equivalence. Since $\mathrm{QCoh}(B^nG)$ is right complete and U^*, U_* preserves filtered colimits, it would be enough to show that $U^*U_*\mathcal{F} \to \mathcal{F}$ is an equivalence for $\mathcal{F} \in \mathrm{QCoh}^{\heartsuit}(B^nG)$. Since $n \geq 2$, and $\mathrm{QCoh}^{\heartsuit}(B^nG)$ is a 1-category, it follows by descent along $\operatorname{Spec} A \to BG$ that $\mathrm{Mod}^{\heartsuit}(A) \simeq \mathrm{QCoh}^{\heartsuit}(B^nG)$. Therefore, in order to show that $U^*U_*\mathcal{F} \to \mathcal{F}$ is an isomorphism for $F \in \mathrm{QCoh}^{\heartsuit}(B^nG)$, by considering filtered colimits, one may assume that $\mathcal{F}$ corresponds to a finitely presented discrete A-module V. By our hypothesis on A, V has finite flat dimension. Using descent along $\operatorname{Spec} A \to B^nG$ and the fact that totalizations of coconnective objects commute with tensors with objects of bounded-above Tor-dimension, we see that $R\Gamma(B^nG, V) \simeq R\Gamma(X, \mathcal{O}) \otimes_A V$; therefore, $R\Gamma(B^nG, V) \otimes_{R\Gamma(X,\mathcal{O})} A \simeq V \otimes_A R\Gamma(X, \mathcal{O}) \otimes_{R\Gamma(X,\mathcal{O})} A \simeq V$. This yields the claim. $\square$

Proposition 3.22. *Under the assumptions of Proposition 3.21, let* $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{QCoh}(B^nG)_{\leq 0}$*. Then the natural map* $\theta_{\mathcal{F}_1,\mathcal{F}_2} : R\Gamma(B^nG, \mathcal{F}_1) \otimes_{R\Gamma(B^nG,\mathcal{O})} R\Gamma(B^nG, \mathcal{F}_2) \to R\Gamma(B^nG, \mathcal{F}_1 \otimes \mathcal{F}_2)$ *is an isomorphism.*

Proof. By filtered colimit considerations, similar to proof of Proposition 3.21, it is enough to prove that $\theta_{\mathcal{F}_1,V_2}$ is an isomorphism, where V_2 is a finitely presented A-module viewed as an object of $\mathrm{QCoh}(B^nG)$ via pullback. Repeating the same argument, it is enough to prove that θ_{V_1,V_2} is an isomorphism, where V_1 is a finitely presented A-module. Note that $V_1 \otimes_A V_2$ has bounded above Tor-dimension. Since totalizations of coconnective objects commute with tensors with objects of bounded-above Tor-dimension, by descent along $\operatorname{Spec} A \to B^nG$, it follows that $R\Gamma(B^nG, V_1 \otimes V_2) \simeq R\Gamma(B^nG, \mathcal{O}) \otimes_A V_1 \otimes_A V_2$. This gives the desired statement. $\square$

[2]Here we use that totalization of coconnective objects commute with filtered colimits

Lemma 3.23. *Let A be a discrete ring such that every finitely presented A-module has finite flat dimension. Let G_1 and G_2 be two flat affine commutative group scheme over* $\operatorname{Spec} A$. *Then for any $n \geq 0$, we have a natural isomorphism $R\Gamma(B^nG_1, \mathcal{O}) \otimes_A R\Gamma(B^nG_2, \mathcal{O}) \simeq R\Gamma(B^n(G_1 \times G_2), \mathcal{O})$.*

Proof. The claim follows directly for $n = 0$. Via induction and the assumption on A, it follows that $R\Gamma(B^nG_i, \mathcal{O})$ has Tor-amplitude in ≤ 0 for $i = 1, 2$ and all $n \geq 0$. To show the desired claim, let us assume that for a given $n \geq 0$, we have natural isomorphism $R\Gamma(B^nG_1, \mathcal{O}) \otimes_A R\Gamma(B^nG_2, \mathcal{O}) \simeq R\Gamma(B^n(G_1 \times G_2), \mathcal{O})$. By descent, we see that

$$R\Gamma(B^{n+1}G_1, \mathcal{O}) \otimes_A R\Gamma(B^{n+1}G_2, \mathcal{O}) \simeq \lim_{[k]\in\Delta} R\Gamma(B^nG_1, \mathcal{O})^{\otimes k} \otimes_A \lim_{[l]\in\Delta} R\Gamma(B^nG_2, \mathcal{O})^{\otimes l}.$$

Using our assumption on A and commuting tensoring of objects with bounded above Tor-dimension with totalization of coconnective objects, the right hand side above is isomorphic to $\lim_{[k]\times[l]\in\Delta\times\Delta} R\Gamma(B^nG_1, \mathcal{O})^{\otimes k} \otimes_A R\Gamma(B^nG_2, \mathcal{O})^{\otimes l}$; which is further isomorphic to

$$\lim_{j\in\Delta} R\Gamma(B^nG_1, \mathcal{O})^{\otimes j} \otimes_A R\Gamma(B^nG_2, \mathcal{O})^{\otimes j},$$

since Δ^{op} is sifted. Now $\lim_{j\in\Delta} R\Gamma(B^nG_1, \mathcal{O})^{\otimes j} \otimes_A R\Gamma(B^nG_2, \mathcal{O})^{\otimes j} \simeq \lim_{j\in\Delta}(R\Gamma(B^nG_1, \mathcal{O}) \otimes_A R\Gamma(B^nG_2, \mathcal{O}))^{\otimes j} \simeq \lim_{j\in\Delta} R\Gamma(B^n(G_1 \times G_2), \mathcal{O})^{\otimes j} \simeq R\Gamma(B^{n+1}(G_1 \times G_2), \mathcal{O})$; thus we are done by induction. ☐

Lemma 3.24. *Let A be a discrete ring such that every finitely presented A-module has finite flat dimension. Let G be a flat affine commutative group scheme over* $\operatorname{Spec} A$. *For any ordinary A-algebra C and any $n \geq 0$, we have a natural isomorphism $R\Gamma(B^nG, \mathcal{O}) \otimes_A C \simeq R\Gamma(B^nG_C, \mathcal{O})$.*

Proof. When $n = 0$, the claim follows diectly since G is flat. Via induction and our assumption on A, it follows that $R\Gamma(B^nG, \mathcal{O})$ has Tor-amplitude in ≤ 0 for all $n \geq 0$. To show the desired claim, let us assume that for a given $n \geq 0$ and any given A-algebra C, we have a natural isomorphism $R\Gamma(B^nG, \mathcal{O}) \otimes_A C \simeq R\Gamma(B^nG_C, \mathcal{O})$. By descent along $\operatorname{Spec} A \to B^{n+1}G$, we have

$$R\Gamma(B^{n+1}G, \mathcal{O}) \otimes_A C \simeq \left(\lim_{[k]\in\Delta} R\Gamma(B^nG, \mathcal{O})^{\otimes k}\right) \otimes_A C.$$

Let us write $C \simeq \operatorname{colim}_i C_i$, where each C_i is a finitely presented A-module. The right hand side above can be rewritten as $\operatorname{colim}\left(\left(\lim_{[k]\in\Delta} R\Gamma(B^nG, \mathcal{O})^{\otimes k}\right) \otimes_A C_i\right)$. By assumption, each C_i has bounded Tor-dimension, so by commutation of totalization of coconnective objects with tensoring with modules of bounded Tor-dimension, we obtain

$$\operatorname{colim}\left(\left(\lim_{[k]\in\Delta} R\Gamma(B^nG, \mathcal{O})^{\otimes k}\right) \otimes_A C_i\right) \simeq \operatorname{colim}\left(\lim_{[k]\in\Delta}\left(R\Gamma(B^nG, \mathcal{O})^{\otimes k} \otimes_A C_i\right)\right).$$

By induction and Lemma 3.23, each term appearing in the totalization in the right hand side is coconnective, and therefore, commutes with filtered colimits. Therefore, the right hand side simplifies to

$$\lim_{[k]\in\Delta}\left(\operatorname{colim}\left(R\Gamma(B^nG, \mathcal{O})^{\otimes k} \otimes_A C_i\right)\right) \simeq \lim_{[k]\in\Delta}\left(\left(R\Gamma(B^nG, \mathcal{O})^{\otimes k} \otimes_A C\right)\right) \simeq \lim_{[k]\in\Delta} R\Gamma(B^nG_C, \mathcal{O})^{\otimes k},$$

where the last step follows from induction. Applying descent along $\operatorname{Spec} C \to B^{n+1}C$, thr right hand term is naturally isomorphic to $R\Gamma(B^{n+1}G_C, \mathcal{O})$, which finishes the proof. ☐

Lemma 3.25. *Let A be a discrete ring such that every finitely presented A-module has finite flat dimension. Let G be a flat affine commutative group scheme over* $\operatorname{Spec} A$ *and let $n \geq 2$ be an integer. Then we have a natural isomorphism*

$$A \otimes_{R\Gamma(B^nG,\mathcal{O})} A \simeq R\Gamma(B^{n-1}G, \mathcal{O})$$

of derived rings.

Proof. We have the following pullback diagram

$$\begin{tikzcd} B^{n-1}G \arrow[r, "f'"] \arrow[d, "g'"] & \operatorname{Spec} A \arrow[d, "g"] \\ \operatorname{Spec} A \arrow[r, "f"] & B^n G. \end{tikzcd}$$

By Proposition 3.21, it suffices to prove that $f^* g_* \mathcal{O} \simeq g'_* f'^* \mathcal{O}$. It would suffice to prove that the $\mathcal{O}$-module pushforward of $\mathcal{O}_A$ along g is quasi-coherent. This follows from faithfully flat descent along $\operatorname{Spec} A \to B^n G$, along with Lemma 3.24. ☐

4. Examples of affine stacks

If k is a field, then a pointed *connected* fpqc sheaf on discrete k-algebras is affine if and only if the homotopy group sheaves are representable by pro-unipotent group schemes, cf. [Toë06, Cor. 2.2.3] or (in characteristic zero) [Lur11b, Prop. 4.4.8]. In this section, we prove a generalization of this result, and also include several additional examples of affine derived k-stacks (not restricting k to be a field) and their corresponding derived rings.

4.1. Generalities.

Proposition 4.1. *Let X be a derived k-stack. Let $k \to k'$ be a map of animated rings. If X is affine, then the base-change $X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ is affine.*

Proof. Follows from Remark 3.11. ☐

Proposition 4.2. *Let X be a derived k-stack. Suppose $k \to k'$ is faithfully flat and the base-change $X \otimes_k k'$ is affine (as a derived k'-stack). Then X is affine.*

Proof. For each n, let $X^{\leq n}$ denote the restriction of X to n-truncated animated k-algebras. It suffices to see that $X^{\leq n}$ is corepresentable by an n-truncated derived ring. Now the constructions that carry k to "sheaves of anima on n-truncated animated k-algebras" and to "n-truncated derived k-algebras" both satisfy flat descent in k. Therefore, testing whether an object of the former belongs to the image of the latter (via the fully faithful embedding) can be done locally in the flat topology, whence the result. ☐

Proposition 4.3. *Let $\mathfrak{X} \to \mathfrak{Y}$ be a map of derived k-stacks. Suppose that:*

(1) $\mathfrak{Y}$ is an affine stack.
(2) There is a map $\operatorname{Spec} A \to \mathfrak{Y}$ with A an animated k-algebra, which is surjective in the fpqc topology, such that the fiber product $\operatorname{Spec} A \times_{\mathfrak{Y}} \mathfrak{X}$ is an affine A-stack.

Then $\mathfrak{X}$ is an affine stack.

Proof. This follows in a way similar to the proof of Proposition 4.2 because the construction which carries an n-truncated $R \in \mathrm{DAlg}_k$ to the ∞-category $\mathrm{DAlg}_{R,\leq n}$ satisfies hyperdescent for the coconnectively faithfully flat topology (Corollary 2.18). ☐

Corollary 4.4. *Let k be a ring, and G an affine group scheme over k. Suppose BG is an affine stack. If X is any affine k-scheme with a G-action, then the quotient stack X/G is affine.* ☐

Remark 4.5 (Relative affineness)**.** Let $\mathfrak{X} \to \mathfrak{Y}$ to be a map of derived k-stacks. One may define $\mathfrak{X}$ to be a relative affine derived stack over Y if for every affine derived scheme $\operatorname{Spec} A$ with a map $\operatorname{Spec} A \to \mathfrak{Y}$, the base change $\operatorname{Spec} A \times_{\mathfrak{Y}} \mathfrak{X}$ is an affine derived stack. For example, (using Corollary 4.8) it follows that $\mathbb{P}^n$ is relatively affine over $B\mathbb{G}_m$.

Example 4.6 (Moduli of formal groups)**.** Let G be the affine group scheme parametrizing formal power series $g(t)$ in one variable such that $g(0) = 0$ and $g'(0) = 1$ (under composition). Then G is a unipotent group scheme and by Proposition 4.11, it follows that BG is an affine stack. Let L denote the Lazard ring. Then G has a natural action on $\operatorname{Spec} L$. By Corollary 4.4, it follows that the quotient stack $\operatorname{Spec} L/G$ is an affine stack. Let $\mathcal{M}_{\mathrm{FG}}$ denote the module stack of formal groups, which is equipped with a natural map $\mathcal{M}_{\mathrm{FG}} \to B\mathbb{G}_m$. The pullback of $\mathcal{M}_{\mathrm{FG}}$ along the fpqc cover $* \to B\mathbb{G}_m$ is the stack $\operatorname{Spec} L/G$, which as we showed, is affine. Therefore, $\mathcal{M}_{\mathrm{FG}}$ is relatively affine over $B\mathbb{G}_m$. See [Man24, Prop. 3.17, Rmk. 3.18].

4.2. **Quasi-affine derived schemes.** In this subsection we show that quasi-affine derived schemes are affine as derived stacks; in fact, we have the following more precise result. Compare [Lur18, Cor. 2.4.2.2] for the analog in spectral algebraic geometry.

Proposition 4.7. *Let X be a quasi-affine derived k-scheme. Then for any animated k-algebra A, we have a natural equivalence*

$$\operatorname{Hom}_{\mathrm{DSch}_k}(\operatorname{Spec} A, X) \simeq \operatorname{Hom}_{\mathrm{DAlg}_k}(R\Gamma(X, \mathcal{O}_X), A).$$

Proof. Suppose X is an open subset of $\operatorname{Spec} B$ for B an animated ring, and is the complement of the closed subset $Z \subset \operatorname{Spec} B$ defined by $x_1, \dots, x_n \in \pi_0(B)$. In fact, by base-change, we have that $R\Gamma(X, \mathcal{O}_X) \in \mathrm{DAlg}_B$ is an idempotent object, i.e., $R\Gamma(X, \mathcal{O}_X) \otimes_B R\Gamma(X, \mathcal{O}_X) = R\Gamma(X, \mathcal{O}_X)$. Thus, the map of anima $\operatorname{Hom}_{\mathrm{DAlg}_k}(R\Gamma(X, \mathcal{O}_X), A) \to \operatorname{Hom}_{\mathrm{DSch}_k}(\operatorname{Spec} A, X) = \operatorname{Hom}_{\mathrm{DAlg}_k}(B, A)$ is an inclusion of a union of connected components and it only remains to determine the image. That is, we need to determine when a map $B \to A$ factors over $R\Gamma(X, \mathcal{O}_X)$. If such a factorization exists, then $A/(x_1, \dots, x_n) = 0$ since $B/(x_1, \dots, x_n) = 0$, which implies that $\operatorname{Spec}(A) \to \operatorname{Spec}(B)$ has image in the open complement of Z. Conversely, if $\operatorname{Spec}(A) \to \operatorname{Spec}(B)$ has image in the open complement of Z, then $B \to A$ factors over $B \to R\Gamma(X, \mathcal{O}_X) \to A$ as desired. □

For the next result, cf. also the discussion preceding [Toë06, Cor. 2.2.11] and [MR23, Ex. 2.1.13.].

Corollary 4.8. *A quasi-affine derived k-scheme is an affine derived k-stack.*

Proof. Follows from Proposition 4.7. □

4.3. **Geometric stacks.** A 0-derived stack is called *geometric* if it can be presented via a flat Hopf algebroid, cf. [Lur18, Def. 9.3.0.1]; for example, this applies to the quotient of an affine scheme by a flat affine group scheme. A key consequence of geometricity is the Tannaka duality results proved in *loc. cit.* and [BHL17].

In this subsection, we describe some examples of geometric stacks which are affine; to prove affineness, we often use Tannakian reconstruction techniques of [Lur18, Ch. 9], following ideas used in [Mon22b].

Construction 4.9. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a presentably symmetric monoidal ∞-category equipped with a right-complete t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ which is compatible with the tensor structure (i.e., $\mathbf{1} \in \mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\geq 0}$ is closed under tensor products). Then the natural symmetric monoidal, cocontinuous functor

$$\otimes \mathbf{1} : \operatorname{Mod}(\operatorname{End}_{\mathcal{C}}(\mathbf{1})) \to \mathcal{C}$$

is right t-exact.

Proposition 4.10. *Let X be locally Noetherian geometric stack. Suppose the functor of Construction 4.9,*

$$\operatorname{Mod}(R\Gamma(X, \mathcal{O}_X)) \to \operatorname{QCoh}(X)$$

exhibits $\operatorname{QCoh}(X)$ *as the left-completion of* $\operatorname{Mod}(R\Gamma(X, \mathcal{O}_X))$ *in the sense of* [Lur18, Prop. C.3.6.3]. *Then X is an affine stack.*

Proof. The functor X on animated k-algebras is the sheafification of its left Kan extension from discrete k-algebras. Thus, it suffices to show that the restriction of X to discrete k-algebras is an affine stack, i.e., belongs to the smallest subcategory of sheaves on discrete k-algebras generated under limits by $K(\mathbb{G}_a, n), n \geq 0$. Using Tannaka duality in the form of [Lur18, Th. 9.5.4.1], we find that for any discrete k-algebra A, we have a fully faithful functor

$$X(A) \simeq \mathrm{Fun}^{\otimes, L}_{\geq 0}(\mathrm{QCoh}(X), \mathrm{Mod}(A))$$

between A-points of X and symmetric monoidal, cocontinuous, right t-exact functors $\mathrm{QCoh}(X) \to \mathrm{Mod}(A)$. Since by assumption $\mathrm{QCoh}(X)$ is the left-completion of $\mathrm{Mod}(R\Gamma(X, \mathcal{O}_X))$ and since $\mathrm{Mod}(A)$ is automatically left-complete, we find by the universal property [Lur18, Prop. C.3.6.3] that cocontinuous, right t-exact functors from $\mathrm{QCoh}(X) \to \mathrm{Mod}(A)$ are identified with such functors from $\mathrm{Mod}(R\Gamma(X, \mathcal{O}_X)) \to \mathrm{Mod}(A)$, and similarly after imposing symmetric monoidal structures; thus we find that

$$X(A) \simeq \mathrm{Fun}^{\otimes, L}(\mathrm{Mod}(R\Gamma(X, \mathcal{O}_X)), \mathrm{Mod}(A)),$$

since any symmetric monoidal cocontinuous functor $\mathrm{Mod}(R\Gamma(X, \mathcal{O}_X)) \to \mathrm{Mod}(A))$ is automatically right t-exact (as the connective objects of the source are generated under colimits and extensions by the unit). But by Morita theory [Lur17, Prop. 7.1.2.7], this is identified with maps of E_∞-algebras $R\Gamma(X, \mathcal{O}_X) \to A$. Writing $R\Gamma(X, \mathcal{O}_X)$ as a colimit of free E_∞-algebras, we find that the functor $A \mapsto X(A)$ on discrete k-algebras A belongs to the subcategory of sheaves of anima generated by the $K(\mathbb{G}_a, n), n \geq 0$, whence is an affine stack. □

Proposition 4.11. *Let k be a ring such that every finitely presented k-module has finite flat dimension. Let $G = \operatorname{Spec} A$ be a flat affine group scheme over k. Suppose every nonzero G-representation in k-modules has a nontrivial fixed vector. Then the stack BG is affine.*

Proof. By Proposition 4.10, it suffices to show that $\mathrm{QCoh}(BG)$ is the left-completion of $\mathrm{Mod}(R\Gamma(BG, \mathcal{O}))$.

For $V \in \mathrm{QCoh}(BG)_{\leq 0}$, we show that the natural map

$$R\Gamma(BG, V) \otimes_{R\Gamma(BG, \mathcal{O})} k \to V \tag{4.3.1}$$

is an equivalence. Note that both sides are exact functors and preserve filtered colimits (on $\mathrm{QCoh}(BG)_{\leq 0}$), so we reduce to the case where $V \in \mathrm{QCoh}(BG)^\heartsuit$ is discrete.

Since we can (by assumption) filter V exhaustively where all the successive subquotients have trivial G-action, we may assume G acts trivially on V and (by passage to filtered colimits) that the underlying k-module of V is finitely presented, and thus has finite Tor-dimension over R by assumption. It follows easily[3] that $R\Gamma(BG, V) = V \otimes_k R\Gamma(BG, \mathcal{O})$ and that (4.3.1) is therefore an equivalence.

Now we claim that $R\Gamma(BG, \mathcal{O}) \to k$ is coconnectively faithfully flat. This follows from Proposition 2.11, noting that the fiber of the map $R\Gamma(BG, \mathcal{O}) \to k$ has Tor-amplitude in degrees ≤ -1 (this fact uses our assumption, cf. Remark 3.19, and the canonical resolution for $R\Gamma(BG, \mathcal{O})$).

It follows that the adjunction $\mathrm{Mod}(R\Gamma(BG, \mathcal{O})) \rightleftarrows \mathrm{QCoh}(BG)$ restricts to an adjunction on coconnective objects, and the right adjoint $R\Gamma(BG, -)$ is fully faithful; since the left adjoint is conservative by coconnective faithful flatness, we find that the adjunction induces an equivalence $\mathrm{Mod}(R\Gamma(BG, \mathcal{O}))_{\leq 0} \simeq \mathrm{QCoh}(BG)_{\leq 0}$, whence $\mathrm{QCoh}(BG)$ is the left-completion $\mathrm{Mod}(R\Gamma(BG, \mathcal{O}))_{\leq 0}$ and the result follows. □

4.4. **Generalization of a result of Toën and Lurie.** If k is a field, then a pointed *connected* fpqc sheaf on discrete k-algebras is affine if and only if the homotopy group sheaves are representable by pro-unipotent group schemes, cf. [Toë06, Cor. 2.2.3] or (in characteristic zero) [Lur11b, Prop. 4.4.8]. It is natural to wonder if this theorem holds over a more general base. However, one direction of this result can already be easily seen to fail over a base ring such as $\mathbb{Z}_p$ (e.g., see [MR23, Ex. 5.4.1]). However, the other direction of this theorem hold over any ring A which has the property that every finitely presented A-module has finite flat dimension (as noted earlier, this includes all regular

[3]Totalizations of coconnective objects commute with tensors with objects of bounded-above Tor-dimension, as one sees by approximation by finite totalizations.

noetherian rings). In this section, we fix a ring A with this property. First, we formulate a definition of unipotent group schemes over A based on the notion of affine stacks.

Definition 4.12. Let A be the ring fixed as before. An affine flat group scheme G over $\operatorname{Spec} A$ is called unipotent if BG is an affine stack.

Remark 4.13. By Proposition 4.11 and Theorem 3.18, the above definition is equivalent to the more classical notion of any nonzero G-representation in A-modules having a nontrivial fixed vector.

Proposition 4.14. *Let G be a flat affine commutative group scheme over a base A fixed as before. Suppose that BG is an affine stack. Then B^nG is an affine stack for all $n \geq 1$.*

Proof. We can assume $n \geq 2$. By induction on n, we may also assume that $B^{n-1}G$ is an affine stack. Let $U(B^mG) := \operatorname{Spec} R\Gamma(B^mG, \mathcal{O})$. By Lemma 3.25, we have a pullback diagram

$$\begin{tikzcd} U(B^{n-1}G) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & U(B^nG). \end{tikzcd}$$

By induction, $B^{n-1}G \simeq U(B^{n-1}G)$. By our assumption on A, the map $R\Gamma(B^nG, \mathcal{O}) \to A$ is coconnectively faithfully flat (cf. proof of Proposition 3.21). By Proposition 3.5, it follows that the map $* \to U(B^nG)$ must be an effective fpqc epimorphism. By descent, along $* \to U(B^nG)$, we see that $\operatorname{colim}_{k \in \Delta^{\mathrm{op}}} (B^{n-1}G)^{\times k} \simeq U(B^nG)$. Applying descent along $* \to B^nG$ to simplify the right hand side of the latter isomorphism, we obtain $B^nG \simeq U(B^nG)$. This proves that B^nG is affine. $\square$

Remark 4.15. The definition of unipotence as in Definition 4.12 is well behaved. For example, it follows that inverse limit of unipotent group schemes (with vanishing $R^1\lim$) is again unipotent. Let $0 \to G' \to G \to G'' \to 0$ be a short exact sequence of unipotent commutative group shemes over A such that G' and G'' are unipotent. Then G is also unipotent. Indeed, one has $BG \simeq BG'' \times_{B^2G'} \operatorname{Spec} A$. Since B^2G' is affine by Proposition 4.14, it follows that BG is affine too, hence the claim.

Now we are ready to prove the following:

Proposition 4.16. *Let A be a base ring fixed as before. Let X be a pointed connected hypercomplete fpqc sheaf of anima on discrete A-algebras. Assume that the homotopy sheaves $\pi_i(X, *)$ are representable by flat unipotent affine group schemes. Then X is an affine stack.*

Proof. Since X is hypercomplete and the fpqc topology is replete, by [MR22, Thm. A], $X \simeq \varprojlim \tau_{\leq n} X$. Since the property of being an affine stack is closed under limits, we can assume that X is n-truncated. We will prove the statement by induction on n. When $n = 0$, the statement is clear since in this case X, being connected, is isomorphic to $\operatorname{Spec} A$. For $n \geq 1$, we suppose that the statement has been established for $(n-1)$-truncated stacks. For an n-truncated pointed connected stack X satisfying the hypothesis of our proposition, we have a pullback diagram

$$\begin{tikzcd} K(\pi_n(X), n) \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r] & \tau_{\leq n-1} X. \end{tikzcd}$$

Since $\pi_n(X)$ is unipotent, by Proposition 4.14, $K(\pi_n(X), n)$ is an affine stack. By induction, $\tau_{\leq n-1} X$ is an affine stack. By Proposition 4.3, it follows that X too, is an affine stack. This finishes the proof. $\square$

4.5. **Duals to formal groups.** Let k be a ring. Let $G = \operatorname{Spec} A$ be a flat, affine group scheme over R which arises as the Cartier dual to a formal group over k. This means in particular that A is a flat commutative, cocommutative Hopf algebra over k such that, as a coalgebra, A is isomorphic to a divided power coalgebra on a finitely generated projective R-module M (cf. [Lur, Sec. 1.1.2]).

It follows that the abelian category of quasi-coherent sheaves on BG is identified with the category of $\mathrm{Sym}^*(M^\vee)$-modules such that each element of $M^\vee$ acts locally nilpotently; the tensor product is the k-linear tensor product. By [Lur18, Cor. 10.4.6.8], $\mathrm{QCoh}(BG)$ is the left-completion of the derived ∞-category of its heart; in this case, we find that $\mathrm{QCoh}(BG) \subset \mathrm{Mod}(\mathrm{Sym}^*_k(M^\vee))$ is the subcategory where the elements of $M^\vee$ act locally nilpotently on homotopy, and the left-completion is redundant.

It follows that $H^*(BG, \mathcal{O})$ is an exterior algebra over k.

Proposition 4.17. *We have* $\mathrm{Mod}(R\Gamma(BG, \mathcal{O})) \simeq \mathrm{QCoh}(BG)$*, compatibly with t-structures. Moreover, the map $R\Gamma(BG, \mathcal{O}) \to k$ is descendable.*

Proof. In fact, the above discussion (and identification) proves the claim, since the unit is a compact generator for $\mathrm{QCoh}(BG)$.

The functor $\mathrm{QCoh}(BG) \to \mathrm{Mod}(k)$ is identified with the forgetful functor from torsion $\mathrm{Sym}^*_k(M^\vee)$-modules to $\mathrm{Mod}(k)$. Descendability amounts to the statement that any sufficiently long composite of maps in $\mathrm{QCoh}(BG)$ whose successive maps pull back to nullhomotopic maps in $\mathrm{Mod}(k)$ is itself nullhomotopic; this assertion now follows from Lemma 4.18. ☐

Lemma 4.18. *Let $M \to M_1 \to \cdots \to M_{n+1}$ be a sequence of maps in* $\mathrm{Mod}(k[x_1, \ldots, x_n])$ *each of which restricts to a nullhomotopic map in* $\mathrm{Mod}(k)$*. Then the composite $M \to M_{n+1}$ is nullhomotopic in* $\mathrm{Mod}(k[x_1, \ldots, x_n])$*.*

Proof. This follows easily using the fact that M admits an $(n+1)$-step filtration in $\mathrm{Mod}(k[x_1, \ldots, x_n])$ whose associated graded terms are obtained by extension of scalars from $\mathrm{Mod}(k)$. ☐

Proposition 4.19. *Let G be a group scheme over k. If G is Cartier dual to a formal group as above, the stack BG is affine.*

Proof. Since BG is a geometric stack such that $\mathrm{QCoh}(BG)$ is compactly generated by the unit (Proposition 4.17), Tannaka reconstruction in the form of [BHL17, Th. 4.1] (or [Lur18, Cor. 9.4.4.7]) identifies $BG(A)$ with the space of cocontinuous symmetric monoidal functors $\mathrm{QCoh}(BG) \to \mathrm{Mod}(A)$ which preserve connective objects. But we have seen that $\mathrm{QCoh}(BG) \simeq \mathrm{Mod}(R\Gamma(BG, \mathcal{O}))$ (with the corresponding t-structure), whence this is identified with E_∞-maps $R\Gamma(BG, \mathcal{O}) \to A$. This proves the affineness claim. ☐

4.6. **Relative prismatization.** In this subsection, we recall some aspects of relative prismatization and a result of Bhatt–Lurie that the relative prismatization of an affine formal scheme is always affine. We use this to deduce an unpublished result of Bhatt that compares quasi-coherent sheaves on the prismatization and modules over prismatic cohomology.

Let (A, I) be a prism in the sense of [BS], and let $\overline{A} = A/I$.

The theory of *loc. cit.* constructs the *derived prismatic cohomology* of animated $\overline{A}$-algebras. This yields a functor $\Delta_{-/A}$ from animated $\overline{A}$-algebras[4] to (p, I)-complete *derived* A-algebras.

Let R be a p-complete animated $\overline{A}$-algebra. The work [BL22b] (see also [Dri20]) defines (via its functor of points) a derived A-stack $\mathrm{WCart}_{\mathrm{Spf}(R)/A}$ on (p, I)-nilpotent animated A-algebras and identifies under mild finiteness hypotheses (on R), $\Delta_{R/A} \simeq R\Gamma(\mathrm{WCart}_{\mathrm{Spf}(R)/A}, \mathcal{O})$, cf. [BL22b, Th. 7.20].

The following result of Bhatt–Lurie shows that the relative prismatization defines an affine derived stack (modulo any power of (p, I)).

[4]The functor $\Delta_{-/A}$ factors through the p-completion.

Theorem 4.20 ([BL22b, Th. 7.17]). *Let R be any p-complete animated $\overline{A}$-algebra. Then for any (p, I)-nilpotent animated A-algebra S, the natural map induces an equivalence*

$$\mathrm{WCart}_{\mathrm{Spf}\, R/A}(S) \simeq \mathrm{Hom}_{\mathrm{DAlg}_A}(\mathbb{\Delta}_{R/A}, S)$$

Quasi-coherent sheaves on $\mathrm{WCart}_{\mathrm{Spf}(R)/A}$ offer a geometric perspective on the ∞-category of (relative) prismatic crystals on $\mathrm{Spf}(R)$, cf. [BL22b, Th. 6.5]. Under mild finiteness assumptions, $\mathrm{QCoh}(\mathrm{WCart}_{\mathrm{Spf}(R)/A})$ can simply identified with (p, I)-complete modules over the prismatic cohomology. This result was explained to us by Bhatt.

Corollary 4.21 (Bhatt). *Let R be any p-complete animated $\overline{A}$-algebra such that the $\overline{A}/p$-module $H_0(L_{R/\overline{A}}/p) = \Omega^1_{R/\overline{A}}/p$ is finitely generated. Then the natural functor induces an equivalence*

$$\mathrm{Mod}(\mathbb{\Delta}_{R/A})_{\widehat{(p,I)}} \simeq \mathrm{QCoh}(\mathrm{WCart}_{\mathrm{Spf}(R)/A}).$$

Proof. It suffices to prove this claim modulo (p, I). In this case, in light of Theorem 4.20, we find that

$$\mathrm{Mod}(\mathbb{\Delta}_{R/A}/(p, I)) \to \mathrm{QCoh}(\mathrm{WCart}_{\mathrm{Spf}(R)/A}/(p, I))$$

exhibits the target as the left-completion of the source, by Theorem 3.18. Thus, it suffices to show that $\mathrm{Mod}(\mathbb{\Delta}_{R/A}/(p, I))$ is left-complete.

Here we will use the criterion of Proposition 2.22: we need to show that there exists a map of derived rings $\mathbb{\Delta}_{R/A}/(p, I) \to B$ which is descendable, where B is connective, and where the relative tensor product $B \otimes_{\mathbb{\Delta}_{R/A}/(p,I)} B$ is connective.

To this end, let $t_1, \ldots, t_n \in R$ be elements whose differentials span $\Omega^1_{R/\overline{A}}/p$. We consider the map $R \to R_\infty = R \otimes_{\overline{A}[x_1,\ldots,x_n]} \overline{A}[x_1^{1/p^\infty}, \ldots, x_n^{1/p^\infty}]$. The map $\mathbb{\Delta}_{R/A} \to \mathbb{\Delta}_{R_\infty/A}$ is descendable modulo (p, I), as shown in [BS, Lem. 8.6] (it is a base-change of a finite coproduct of copies of the map $\mathbb{\Delta}_{\overline{A}[x]/A} \to \mathbb{\Delta}_{\overline{A}[x^{1/p^\infty}]/A}$). Moreover, $\mathbb{\Delta}_{R_\infty/A}$ and $\mathbb{\Delta}_{R_\infty \otimes_R R_\infty/A}$ are connective since $R_\infty, R_\infty \otimes_R R_\infty$ have vanishing $\Omega^1_{-/\overline{A}}/p$, in light of the Hodge–Tate comparison. $\square$

5. p-adic homotopy theory

In this section, we discuss how aspects of p-adic homotopy theory, especially as in [Lur13], can be formulated in the language of affine stacks (see Theorem 5.22).

Let $\mathcal{S}_{p-\mathrm{fin}} \subset \mathcal{S}$ denote the ∞-category of spaces X which are *p-finite:* that is, X has finitely many connected components, X is r-truncated for some r, and all the homotopy groups at any connected component are p-groups. The starting point for this subsection is to prove the following basic result.

Theorem 5.1. *Let k be an algebraically closed field of characteristic p. The singular cochains functor $C^*(-; k)$ embeds the ∞-category $\mathcal{S}_{p-\mathrm{fin}}$ contravariantly fully faithfully into the ∞-category DAlg_k. Moreover, the embedding carries finite limits in $\mathcal{S}_{p-\mathrm{fin}}$ to finite colimits in DAlg_k, and its essential image is contained inside the compact objects of DAlg_k.*

Versions of Theorem 5.1 appear in [Kř93, Goe95], and (for E_∞-algebras instead of derived rings) in [Man01]. In [Lur13], Lurie formulates and proves various generalizations of Theorem 5.1 where k is replaced by an arbitrary $\mathbb{F}_p$-algebra,[5] $\mathcal{S}_{p-\mathrm{fin}}$ is replaced by "p-constructible étale sheaves on $\mathrm{Spec}(k)$" (or more generally pro-objects therein), in the setting of E_∞-algebras over k.

The purpose of this subsection is to explain a derived k-algebra version of the results in [Lur13] using affine stacks.[6]

To warm up, let us sketch a proof of Theorem 5.1 in the present language, which will contain many of the needed ingredients. The following remark will be useful.

[5] In fact, [Lur13] works with more general E_∞-rings.

[6] The connection between the theory of affine stacks and p-adic homotopy theory, at least over a separably closed field, already appears in [Toë06].

Remark 5.2 (Cogenerators for $\mathcal{S}_{p-\mathrm{fin}}$)**.** The subcategory $\mathcal{S}_{p-\mathrm{fin}} \subset \mathcal{S}$ is closed under finite limits, and it is the smallest subcategory closed under finite limits which contains the $K(\mathbb{Z}/p, n), n \geq 2$. In fact, this subcategory contains any finite coproduct of copies of $K(\mathbb{Z}/p, n), n \geq 2$ (where we use $\Omega^2 K(\mathbb{Z}/p, 2)$ is a finite discrete set). The claim then follows from [Lur13, Lem. 2.4.16].

Proof of Theorem 5.1. Given a space X, we consider the sheafification of the constant presheaf X on the site of all k-algebras equipped with the *étale* topology (i.e., the big étale site of k). This gives a fully faithful functor

$$\pi^* : \mathcal{S} \to \mathrm{Shv}(k - \mathrm{alg}_{\mathrm{et}}, \mathcal{S}),$$

which preserves finite limits. Note that affine stacks form a full subcategory of the target. We claim that the functor π^* carries p-finite spaces into affine stacks. Now $\mathcal{S}_{p-\mathrm{fin}}$ is the smallest subcategory of $\mathcal{S}$ closed under finite limits which contains the spaces $K(\mathbb{Z}/p, n), n \geq 2$ (Remark 5.2). Since the class of affine stacks is closed under finite coproducts and arbitrary limits, it suffices to show that $\pi^*(K(\mathbb{Z}/p, n))$ is an affine stack for any $n \geq 2$; however, this follows from the Artin–Schreier sequence, which gives a fiber sequence $K(\mathbb{Z}/p, n) \to K(\mathbb{G}_a, n) \to K(\mathbb{G}_a, n)$. It is easy to see that $R\Gamma(\pi^*(-), \mathcal{O}) = C^*(-; k)$, whence the full faithfulness follows in light of Corollary 3.8.

Finally, we need to show that if $X \in \mathcal{S}_{p-\mathrm{fin}}$, then $C^*(X, k)$ is compact as a derived k-algebra, and that the construction $X \mapsto C^*(X, k)$ preserves finite limits. Here we will use frequently that pullbacks of affine k-stacks correspond to pushouts in the ∞-category of coconnective derived k-algebras (Corollary 3.8).

First, the fiber sequence $K(\mathbb{Z}/p, n) \to K(\mathbb{G}_a, n) \to K(\mathbb{G}_a, n)$ of affine stacks shows (for $n \geq 1$) leads to a square of derived k-algebras

$$\begin{tikzcd}
\mathrm{Sym}^*(k[-n]) \arrow[r] \arrow[d] & \mathrm{Sym}^*(k[-n]) \arrow[d] \\
k \arrow[r] & C^*(K(\mathbb{Z}/p, n), k)
\end{tikzcd}$$

which is cocartesian in coconnective derived k-algebras. Since the pushout in all derived k-algebras is automatically coconnective (by Corollary 2.14), it follows that the above square is also a pushout in DAlg_k, whence $C^*(K(\mathbb{Z}/p, n), k)$ is a compact object of DAlg_k.

Now let us check that pullbacks in $\mathcal{S}_{p-\mathrm{fin}}$ are carried to pushouts in DAlg_k. Since π^* preserves pullbacks, it follows that pullbacks in $\mathcal{S}_{p-\mathrm{fin}}$ are carried to pushouts in $\mathrm{DAlg}_{k,\leq 0}$. Thus, it suffices to show that if $X_1 \to Y, X_2 \to Y$ are maps in $\mathcal{S}_{p-\mathrm{fin}}$, then the relative tensor product $C^*(X_1, k) \otimes_{C^*(Y,k)} C^*(X_2, k)$ is coconnective; however, this follows because the Frobenius on $C^*(X_1, k) \otimes_{C^*(Y,k)} C^*(X_2, k)$ is an isomorphism, and the Frobenius annihilates the higher homotopy groups of an animated ring [BS17, Rem. 11.8]. By Remark 5.2 we are done. $\square$

Next, we recall some aspects of the theory of *p-constructible sheaves* as developed in [Lur13, Sec. 2.3].

Definition 5.3 (p-constructible sheaves, cf. [Lur13, Def. 2.3.1])**.** Let X be a qcqs scheme. We say that a sheaf of spaces $\mathcal{F}$ on X_{et} is *p-constructible* if:

(1) $\mathcal{F}$ is n-truncated for some n.
(2) $\mathcal{F}$ is a coherent object (see [Lur18, Appx. A]) of the ∞-topos $\mathrm{Shv}(X_{\mathrm{et}}, \mathcal{S})$.
(3) For every geometric point $x : \operatorname{Spec} k \to X$, the stalk $x^*\mathcal{F}$ is a p-finite space.

If $X = \mathrm{Spec}(\pi_0 A)$ for A a p-nilpotent E_∞-algebra satisfying mild conditions, then in [Lur13, Cor. 2.6.12], a fully faithful contravariant embedding from p-constructible sheaves on X into E_∞-algebras over A is constructed. We will prove an analog of this result in the context of derived $\mathbb{F}_p$-algebras. First, we need an analog of Remark 5.2 in this more general context, which we will deduce from the structural results in [Lur13].

Proposition 5.4. *Let R be a commutative ring. Then the class of p-constructible objects in* $\mathrm{Shv}((\operatorname{Spec} R)_{\mathrm{et}}, \mathcal{S})$ *is the smallest class containing $K(\mathcal{F}, n)$ for $\mathcal{F}$ a constructible sheaf of $\mathbb{F}_p$-vector spaces on* $\operatorname{Spec} R$ *and closed under finite limits.*

Proof. We denote $X = \operatorname{Spec} R$. Let us first prove the result in the case where $\mathcal{F}$ is *locally constant*: that is, when there exists a finite Galois cover $f : X' \to X$ such that $f^*\mathcal{F}$ is the constant sheaf associated to a p-finite space. In this case, we can resolve $\mathcal{F}$ by a (finite) totalization of $f_* f^* \mathcal{F}, f_* f^* f_* f^* \mathcal{F}, \dots$ of pushforwards of *constant* sheaves on X' (with values a p-finite space). For constant sheaves, the claim follows from Remark 5.2, so $f^*\mathcal{F}$ belongs to the smallest subcategory of $\mathrm{Shv}(X')$ generated under finite limits by $K(\mathcal{A}, n)$ for constant sheaf of finite dimensional $\mathbb{F}_p$-vector spaces $\mathcal{A}$ on X'; pushing forward, we obtain the desired claim in this case.

Now let us treat the general case. By [Lur13, Th. 2.3.24], there exists a finite stratification by closed subschemes

$$X = X_0 \supset X_1 \supset X_2 \supset X_3 \supset \cdots \supset X_{n+1} = \emptyset$$

such that each X_i has quasi-compact open complement in X, and such that $\mathcal{F}|_{X_i \setminus X_{i+1}}$ is locally constant.

We have already proved the result when $n = 0$ (so that $\mathcal{F}$ itself is locally constant), and in general we prove it by induction on n. By the inductive hypotheses, we may assume that there exists a closed subscheme $i : Z \subset X$ with quasi-compact open complement $j : U \subset X$ such that

1. $i^*\mathcal{F} \in \mathrm{Shv}(Z)$ belongs to the subcategory generated under finite limits by $K(\mathcal{F}, n)$, where $\mathcal{F}$ is a constructible sheaf of $\mathbb{F}_p$-vector spaces on Z.
2. $j^*\mathcal{F} \in \mathrm{Shv}(U)$ becomes constant (with values in p-finite spaces) after pullback along a finite Galois cover.

We have a pullback square in $\mathrm{Shv}(X)$,

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & i_* i^* \mathcal{F} \arrow[d] \\ j_* j^* \mathcal{F} \arrow[r] & i_* i^* j_* j^* \mathcal{F} \end{tikzcd}$$

Note that the bottom arrow only depends on $j^*\mathcal{F}$. By induction, we may assume that $j^*\mathcal{F} \in \mathrm{Shv}(U)$ belongs to the subcategory generated under finite limits by $K(\mathcal{A}, n)$, for $n \geq 0$, where $\mathcal{A}$ is a constructible sheaf of $\mathbb{F}_p$-vector spaces. Thus, $\mathcal{F}$ belongs to the subcategory of $\mathrm{Shv}(X)$ generated by pullbacks of diagrams

$$\begin{tikzcd} & i_* i^* \mathcal{F} \arrow[d] \\ K(j_* \mathcal{A}, n) \arrow[r] & K(i_* i^* j_* \mathcal{A}, n) \end{tikzcd}$$

for various $n \geq 0$. Using the pulback diagram

$$\begin{tikzcd} K(j_* \mathcal{A}, n) \arrow[r] \arrow[d] & K(i_* i^* j_* \mathcal{A}, n) \arrow[d] \\ 0 \arrow[r] & K(j_! \mathcal{A}, n+1) \end{tikzcd}$$

the result now follows. □

Proposition 5.5. *Let R be an $\mathbb{F}_p$-algebra, and $X = \operatorname{Spec}(R)$. Let $\mathcal{F} \in \mathcal{D}^b_{\mathrm{cons}}(X_{\mathrm{et}}, \mathbb{F}_p)$. The pullback of $\mathcal{F}$ to the big étale site of X belongs to the thick subcategory generated by $\mathbb{G}_a$.*

Proof. By [Sta24, Tag 09YU], we may assume that R is Noetherian. Further, the constructible derived ∞-category $\mathcal{D}^b_{\mathrm{cons}}(X_{\mathrm{et}}, \mathbb{F}_p)$ is generated as a thick subcategory by the objects $f_*(\mathbb{F}_p)$, for

$f : Y \to X$ a finite, finitely presented morphism, cf. [Mat22, Lem. 6.19]. Thus, we may assume $\mathcal{F} = f_*(\mathbb{F}_p)$. Suppose $Y = \operatorname{Spec}(R[x_1, \ldots, x_n]/(v_1, \ldots, v_m))$. We set $S := R[x_1, \ldots, x_n]/(v_1, \ldots, v_m)$. We define $S' := \bigotimes_{i=1}^m \operatorname{cofib}(R[x_1, \ldots, x_n] \xrightarrow{v_i} R[x_1, \ldots, x_n])$, where the (derived) tensor product is taken over $R[x_1, \ldots, x_n]$. Then S' is an animated $R[x_1, \ldots, x_n]$-algebra such that $\pi_0(S') \simeq S$. We claim that S' is a perfect complex over R. Note that S' has finite Tor-dimension as an $R[x_1, \ldots, x_n]$-module, and therefore, also as an R-module. Thus it suffices to prove that S' is pseudo-coherent. Using Koszul complexes, and the fact that $R[x_1, \ldots, x_n]$ is Noetherian, one sees that $\pi_i(S') = 0$ for $i > m$ and $\pi_i(S')$ is a finite $R[x_1, \ldots, x_n]$-module for all i. Thus it follows that $\pi_i(S')$ is also finite as an S-module. Since S is finite as an R-module, $\pi_i(S')$ is a finite R-module. Since R is Noetherian, by [Sta24, Tag 066E], S' is pseudo-coherent, and thus, is a perfect complex over R, as claimed.

Since S' is a perfect complex over R, using the Artin-Schreier sequence, we see that the functor that sends an R-algebra R' to $R\Gamma_{\mathrm{et}}(S' \otimes_R R', \mathbb{F}_p)$ belongs to the thick subcategory generated by $\mathbb{G}_a$. Using the fact that for any animated $\mathbb{F}_p$-algebra, Frobenius induces the zero map on π_i for $i > 0$, and the Artin-Schreier sequence, it follows that $R\Gamma_{\mathrm{et}}(S' \otimes_R R', \mathbb{F}_p) \simeq R\Gamma_{\mathrm{et}}(S \otimes_R R', \mathbb{F}_p)$. This proves the proposition. ☐

Proposition 5.6. *Let A be an $\mathbb{F}_p$-algebra and $X = \operatorname{Spec}(A)$. Let $\mathcal{F} \in \operatorname{Shv}_{p\text{-cons}}(X_{\mathrm{et}}, \mathcal{S})$. Then the pullback of $\mathcal{F}$ to the big étale site is an affine A-stack. Moreover, it belongs to the subcategory of affine A-stacks generated under finite limits and retracts by the $K(\mathbb{G}_a, n)$, for $n \geq 0$.*

Proof. By Proposition 5.4, it suffices to consider the case when $\mathcal{F} = K(\mathcal{A}, n)$ for $\mathcal{A}$ being a constructible abelian sheaf of $\mathbb{F}_p$-vector spaces on X. Then, the result follows from Proposition 5.5. ☐

Our next result is an analog of the correspondence for E_∞-rings that is proved in [Lur13, Cor. 2.6.12].

Definition 5.7. Let $X = \operatorname{Spec} A$ for an $\mathbb{F}_p$-algebra A and $\mathcal{F} \in \operatorname{Shv}_{p\text{-cons}}(X_{\mathrm{et}}, \mathcal{S})$. We define $C^*(\mathcal{F}, O)$ to be the derived ring of global sections of $\mathcal{F}$ viewed as an affine stack.

Theorem 5.8. *The ∞-category of $\operatorname{Shv}_{p\text{-cons}}(X_{\mathrm{et}}, \mathcal{S})$ contravariantly embeds fully faithfully into DAlg_A, via the embedding sending a sheaf $\mathcal{F}$ to $C^*(\mathcal{F}, \mathcal{O})$. This embedding carries finite limits in $\operatorname{Shv}_{p\text{-cons}}(X_{\mathrm{et}}, \mathcal{S})$ to finite colimits in DAlg_A. Moreover, for any such $\mathcal{F}$, the derived algebra $C^*(\mathcal{F}, \mathcal{O})$ is a compact object of DAlg_A.*

Proof. We follow the strategy of the proof of Theorem 5.1.

Namely, we can (fully faithfully) pull back $\operatorname{Shv}_{p\text{-cons}}(X_{\mathrm{et}}, \mathcal{S})$ into sheaves on the big étale site of X; this embedding preserves finite limits, and we have seen (Proposition 5.6) it has image inside affine stacks. Consequently, we obtain the fully faithful embedding into DAlg_A as stated in the theorem.

Given p-constructible sheaves $\mathcal{F}_1, \mathcal{F}_2 \to \mathcal{G}$, we have a map of derived k-algebras,

$$C^*(\mathcal{F}_1, \mathcal{O}) \otimes_{C^*(\mathcal{G}, \mathcal{O})} C^*(\mathcal{F}_2, \mathcal{O}) \to C^*(\mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2, \mathcal{O}). \tag{5.0.1}$$

We know that this map becomes an equivalence after applying $\widetilde{\tau_{\leq 0}}$, since pullbacks of affine A-stacks map to pushouts of coconnective derived A-algebras. Thus, it suffices to show that the pullback on the left-hand-side remains coconnective.

Suppose first that A is a polynomial $\mathbb{F}_p$-algebra (potentially on infinitely many generators). Then each of the derived A-algebras $C^*(\mathcal{F}_1, \mathcal{O}), C^*(\mathcal{G}, \mathcal{O}), C^*(\mathcal{F}_2, \mathcal{O})$ can be written as totalizations of cosimplicial A-algebras each of whose terms is an étale A-algebra. Therefore, these derived $\mathbb{F}_p$-algebras become perfect after base-change along the faithfully flat map $A \to A_{\mathrm{perf}}$. Therefore, the derived A-algebra $C^*(\mathcal{F}_1, \mathcal{O}) \otimes_{C^*(\mathcal{G}, \mathcal{O})} C^*(\mathcal{F}_2, \mathcal{O})$ becomes perfect after base-change along $A \to A_{\mathrm{perf}}$ and is therefore coconnective by [BS17, Rem. 11.8]. In this case, we moreover have that both sides have Tor-amplitude in degrees ≤ 0 over A.

The general case now reduces to the case of a polynomial algebra, by choosing a surjection from a polynomial algebra and base-changing all terms involved (noting the Tor-amplitude ≤ 0 property remarked in the last paragraph).

The compactness now follows from Proposition 5.6, which shows that for an $\mathbb{F}_p$-étale sheaf $\mathcal{A}$, we have that $K(\mathcal{A}, n)$ for $n \gg 0$ is compact as a derived A-algebra. □

After passage to pro-objects, Theorem 5.8 extends and yields a fully faithful embedding, as follows.

Proposition 5.9. *Let A be an $\mathbb{F}_p$-algebra and $X = \mathrm{Spec}(A)$. Then the pullback functor induces a fully faithful, cocontinuous embedding $C^*(-, \mathcal{O}) : \mathrm{Pro}(\mathrm{Shv}_{p\text{-cons}}(X_{\mathrm{et}}, \mathcal{S}))^{\mathrm{op}} \subset \mathrm{DAlg}_A$.* □

In the remainder of the section, we precisely identify the image of Proposition 5.9 in the case of a *regular* $\mathbb{F}_p$-algebra using results of Emerton–Kisin, [EK04]. We need to first recall some facts about Frobenius modules.

Definition 5.10 (Frobenius modules)**.** Let A be an $\mathbb{F}_p$-algebra. A *Frobenius module* over A is an A-module M equipped with a map $F : \varphi^* M \to M$, for φ^* the base-change functor along the Frobenius $\varphi : A \to A$. A Frobenius module can also be regarded as a left module over the twisted polynomial ring $A[F]$ (where the commutation is $Fa = a^p F$). We will consider the derived ∞-category of Frobenius modules, or of the ring $A[F]$, denoted $\mathcal{D}(A[F])$.

Example 5.11. Let R be a derived A-algebra. Then the homotopy groups of R are Frobenius modules over A, via the internal Frobenius endomorphism of derived $\mathbb{F}_p$-algebras $R \to R$ (which is semilinear over the Frobenius of A).

In the following, we need the following upgrade of the above example.

Construction 5.12 (From derived algebras to Frobenius modules)**.** Note that we have a forgetful functor $\mathrm{DAlg}_A \to \mathcal{D}(A[F])$. We will denote its left adjoint as

$$\mathrm{free}_{A[F]} : \mathcal{D}(A[F]) \to \mathrm{DAlg}_A.$$

Example 5.13. Considering $\mathbb{F}_p$ as a Frobenius module over itself, it follows from the definition that $\mathrm{free}_{\mathbb{F}_p[F]} \mathbb{F}_p \simeq \mathbb{F}_p[x]/(x^p - x)$.

Definition 5.14 (Finitely generated unit Frobenius modules)**.** A Frobenius module M over A is said to be *finitely generated unit* if:

(1) The map $F : \varphi^* M \to M$ is an isomorphism.
(2) M is finitely generated as an $A[F]$-module.

If only (1) is assumed, then M is said to be *unit.*

An object of $\mathcal{D}(A[F])$ is said to be *finitely generated unit* if it is bounded and the homology groups are finitely generated unit. We let $\mathcal{D}_{\mathrm{fgu}}(A[F]) \subset \mathcal{D}(A[F])$ be the subcategory of finitely generated unit objects.

The class of finitely generated unit $A[F]$-modules is studied by Lyubeznik [Lyu97]. It is an abelian subcategory of the category of $A[F]$-modules, and is closed under subobjects inside the (also abelian) category of unit $A[F]$-modules. We refer to [BL19, Sec. 11.3] for an account of the subcategory $\mathcal{D}_{\mathrm{fgu}}(A[F]) \subset \mathcal{D}(A[F])$ (and actually a generalization when A is not regular). It is shown in *loc. cit.* that any object of $\mathcal{D}_{\mathrm{fgu}}(A[F])$ is compact as an object of $\mathcal{D}(A[F])$.

Definition 5.15 (Solvable Frobenius modules)**.** A Frobenius module M over A is said to be *solvable* if it is a filtered colimit of finitely generated unit $A[F]$-modules. An object of $\mathcal{D}(A[F])$ is said to be *solvable* if the homology groups are solvable $A[F]$-modules. A derived A-algebra R is said to be *solvable* if the underlying object of $\mathcal{D}(A[F])$ is solvable.

Proposition 5.16. *(1) The category of solvable $A[F]$-modules is an abelian subcategory of the category of unit $A[F]$-modules which is closed under subobjects, quotients, and extensions.*

(2) *Any bounded-above object of* $\mathcal{D}_{\mathrm{solv}}(A[F])$ *can be written as a filtered colimit of objects in* $\mathcal{D}_{\mathrm{fgu}}(A[F])$.

Proof. Part (1) easily follows from the corresponding results for finitely generated unit modules. For part (2), since $\mathcal{D}_{\mathrm{fgu}}(A[F])$ consists of compact objects in $\mathcal{D}(A[F])$, it suffices to show that any object of $\mathcal{D}_{\mathrm{solv}}(A[F])$ belongs to the localizing subcategory generated by $\mathcal{D}_{\mathrm{fgu}}(A[F])$, which follows from the constructions. □

Example 5.17. Let A be a regular $\mathbb{F}_p$-algebra and A' be an étale A-algebra. Then A' (with its Frobenius action) is a finitely generated unit $A[F]$-module. This is a very special case of the following general result.

Theorem 5.18 (Emerton–Kisin [EK04])**.** *Let* $X = \mathrm{Spec}(A)$ *for a regular* $\mathbb{F}_p$*-algebra* A*. The functor* $\mathrm{RHom}(-, \mathbb{G}_a)$ *establishes a fully faithful, contravariant embedding from* $\mathcal{D}^b_{\mathrm{cons}}(X, \mathbb{F}_p)$ *into* $\mathcal{D}(A[F])$*, with image exactly the finitely generated unit objects.*

Proposition 5.19. *Let* $X = \mathrm{Spec}(A)$ *for a regular* $\mathbb{F}_p$*-algebra* A*. If* $\mathcal{F} \in \mathrm{Shv}_{p\text{-cons}}(X)$*, then* $C^*(\mathcal{F}, \mathcal{O}) \in \mathrm{DAlg}_A$ *is solvable.*

Proof. This follows because $C^*(\mathcal{F}, \mathcal{O})$ can be written as the totalization of a cosimplicial object in étale A-algebras (by coherence of $\mathcal{F}$), and any étale A-algebra defines a finitely generated unit $A[F]$-module. □

Proposition 5.20. *Let* $X = \mathrm{Spec}(A)$ *for a regular* $\mathbb{F}_p$*-algebra* A *and* $M \in \mathcal{D}_{\mathrm{fgu}}(A[F])$*. Then the derived* A*-algebra* $\mathrm{free}_{A[F]}(M)$ *can be written as* $C^*(\mathcal{F}, \mathcal{O})$ *for some* p*-constructible sheaf* $\mathcal{F}$ *on* X*. In particular,* $\mathrm{free}_{A[F]}(M)$ *is solvable by Proposition 5.19.*

Proof. First, $\mathrm{free}_{A[F]}(M)$ base-changed along $A \to A_{\mathrm{perf}}$ to a derived A_{perf}-algebra which is easily seen to be perfect, whence coconnective; here we also use that $A \to A_{\mathrm{perf}}$ is faitfully flat by Kunz's theorem since A is regular.

Now let us consider the affine A-stack corepresented by $\mathrm{free}_{A[F]}(M)$. By definition, this affine stack is given by the functor which sends an A-algebra A' to $\Omega^\infty \mathrm{RHom}_{A[F]}(M, A')$. By the mod p Riemann–Hilbert correspondence of [EK04] (cf. the last two paragraphs of the proof of [Mat22, Th. 6.20]) it follows that this functor is $\Omega^\infty(\pi^* \mathcal{F})$ for some $\mathcal{F} \in \mathcal{D}^b_{\mathrm{cons}}(X, \mathbb{F}_p)$. In particular, the affine stack corresponds to a p-constructible sheaf on X, whence the result. □

Remark 5.21. The above argument does not use the full strength of the results of [EK04]. Rather, one uses that if M is a finitely generated unit Frobenius module, then the construction carrying an A-algebra A' to $\mathrm{RHom}_{A[F]}(M, A')$ is the pullback from the small étale site to the big étale site of an object of $\mathcal{D}^b_{\mathrm{cons}}(X, \mathbb{F}_p)$. This can be checked directly with a henselian rigidity argument (as is done in [Mat22, Th. 6.20]).

Theorem 5.22. *Let* $X = \mathrm{Spec}(A)$ *for a regular* $\mathbb{F}_p$*-algebra* A*. Then the fully faithful embedding of Proposition 5.9,*

$$C^*(-, \mathcal{O}) : \mathrm{Pro}(\mathrm{Shv}_{p\text{-cons}}(X_{\mathrm{et}}, \mathcal{S}))^{\mathrm{op}} \subset \mathrm{DAlg}_A$$

has image precisely the solvable A*-algebras.*

Proof. Using the bar resolution to resolve a derived A-algebra by free algebras (on underlying $A[F]$-modules) and Proposition 5.20, we see that it suffices to show that $\mathrm{free}_{A[F]}(M)$ belongs to the essential image of $C^*(-, \mathcal{O})$ for $M \in \mathcal{D}(A[F])$ solvable and coconnective. By passage to filtered colimits (Proposition 5.16), we reduce to the case where $M \in \mathcal{D}_{\mathrm{fgu}}(A[F])$, which we have proved in Proposition 5.20. □

In the case of a separably closed field, the analog of this result for E_∞-rings is established in [Lur13, Th. 3.5.8].

6. Formal groups and derived rings

In this section, we will establish certain equivalence of categories between certain class of augmented derived rings and formal lie groups. Using these, we explain a different construction of a 1-dimensional formal group constructed by Drinfeld in [Dri21].

Proposition 6.1. *Let A be a discrete ring. Let $\mathcal{F}_A$ denote the ∞-category of augmented derived rings R over A such that $\pi_1(\operatorname{Spec} R)$ is an abelian sheaf, $H^1(R)$ is a projective module of rank r over A, and $H^*(R) \simeq \wedge^* H^1(R)$. Then $\mathcal{F}_A$ is equivalent to the category of formal lie groups of dimension r via a functor that sends a formal lie group*

$$F \mapsto R\Gamma(BF^\vee, \mathcal{O}).$$

Proof. Let $R \in \mathcal{F}_A$. We will study the tensor product $A \otimes_R A$. Using the bar resolution, we obtain the following simplicial object in $D(A)$

$$\cdots R \otimes_A R \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} R \rightrightarrows A \tag{6.0.1}$$

whose colimit is the tensor product $A \otimes_R A$. Note that the terms of (6.0.1) are coconnective and have compatible increasing exhaustive $\mathbf{N}$-indexed filtrations coming from the truncation functors. By taking colimit, we obtain an increasing exhaustive $\mathbf{N}$-indexed filtration on $A \otimes_R A$, which will be denoted by Fil^n. One sees that the n-th graded piece gr^n of this filtration is given by $B_n[-n]$, where $B_0 \simeq A$ and for $n \geq 1$, B_n is the colimit of the following simplicial object

$$C_n := \cdots H^n(R^{\otimes 3}) \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} H^n(R^{\otimes 2}) \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} H^n(R) \rightrightarrows 0\ .$$

We note that $V := H^1(R)$ is a projective module over A of rank r and the simplicial object C_1 is of the form

$$\cdots V^{\oplus 3} \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} V^{\oplus 2} \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} V \rightrightarrows 0\ .$$

Considering V as an abelian group, we observe that the above simplicial object is isomorphic to the classifying object BV.

Since $H^*(R) \simeq \wedge^* H^1(R)$, it also follows that $H^*(R^{\otimes k}) \simeq \wedge^* H^1(R^{\otimes k})$. Thus we see that C_n is computed by applying $\wedge^n$ termwise to the simplicial object C_1. Since $C_1 \simeq BV$, it follows that $B_n \simeq \wedge^n(V[1])$. By the décalage formulas, we have $\wedge^n(V[1]) \simeq (\Gamma^n V)[n]$. Therefore, $\mathrm{gr}^n \simeq B_n[-n] \simeq \Gamma^n V$. In particular, gr^n is discrete. Therefore, we conclude that $A \otimes_R A$ is also discrete. Repeating the same argument and base changing along $A \to A/I$ for any ideal $I \subset A$, we can also conclude that $A \otimes_R A$ is flat over A. Since $A \otimes_R A$ has the structure of a Hopf algebra, $G := \operatorname{Spec}(A \otimes_R A)$ is a flat group scheme over A.

Lemma 6.2. *In the above set up,* $\operatorname{Spec} R \simeq BG$ *as pointed affine stacks. In particular, BG is an affine stack.*

As a consequence of the above lemma and Theorem 3.6, it follows that the ∞-category $\mathcal{F}_A$ embeds into the category of pointed stacks of the form BH, for some affine group scheme H. However, the latter is a 1-category; therefore, $\mathcal{F}_A$ is a 1-category as well.

Now we proceed onto proving that dual of $G = \operatorname{Spec}(A \otimes_R A)$ is a formal lie group. By Lemma 6.2, $G \simeq \pi_1(\operatorname{Spec} R)$; therefore G is commutative by our hypothesis. In other words, $A \otimes_R A$ is a cocommutative Hopf algebra equipped with a filtration Fil^n stable under comultiplication whose associated graded is $\oplus_{n \geq 0} \Gamma^n V$. Let $B := \operatorname{Hom}_A(A \otimes_R A, A)$. Then B is a commutative A-algebra Let $J_n \subset B$ be the set of A-module maps $f : A \otimes_R A \to A$ such that $f(x) = 0$ for $x \in \mathrm{Fil}^n$. By design, J_n is the kernel of the surjective map $B \to (\mathrm{Fil}^n)^*$. Because Fil^n is stable under comultiplication, it follows that J_n is an ideal of B and $B \simeq \varprojlim_n B/J_n$. Note that we have an exact sequence $0 \to J_2 \to J_1 \to J_1/J_2 \to 0$. By construction, it follows that $J_1/J_2 \simeq V^*$, where $V = H^1(R)$. Since V is prorjective, the latter exact sequence must split and we obtain a map $J_1/J_2 \hookrightarrow J_1 \hookrightarrow B$ of A-modules. Since B is a commutative A-algebra, we obtain a map

$\mathrm{Sym}^*_A(J_1/J_2) \to B$. This map is filtered with respect to the natural decreasing filtrations on both sides. Since $J_n/J_{n+1} \simeq (\Gamma^n V)^* \simeq \mathrm{Sym}^n V^*$, it follows that the latter map induces isomorphism on the graded pieces. Since B is complete with respect to the filtration, it follows that we have an isomorphism $(\mathrm{Sym}^*(V^*))^\wedge \xrightarrow{\simeq} B$.

Thus, we obtain a functor from $\mathcal{F}_A$ to the category of formal lie groups which is an inverse to the functor described in the proposition. $\square$

Proof of Lemma 6.2. Note that $\mathrm{Spec}\, R$ is automatically pointed by the map $\mathrm{Spec}\, A \to \mathrm{Spec}\, R$ that is induced by the augmentation of R. Since $G \simeq \mathrm{Spec}\,(A \otimes_R A)$, by virtue of Cech descent, it would be enough to show that $\mathrm{Spec}\, A \to \mathrm{Spec}\, R$ is an fpqc epimorphism. By Proposition 2.11, one knows that the map $R \to A$ is coconnectively faithfully flat. By Proposition 3.5, it follows that $\mathrm{Spec}\, A \to \mathrm{Spec}\, R$ is an fpqc epimorphism, which finishes the proof of the lemma. $\square$

Proposition 6.3. *Let A be a discrete ring. Let $\mathcal{G}_A$ denote the ∞-category of augmented derived rings R over A such that $H^2(R)$ is a projective module of rank r over A, and $H^*(R) \simeq \mathrm{Sym}^* H^2(R)$. Then $\mathcal{G}_A$ is equivalent to the category of formal lie groups of dimension r via a functor that sends a formal lie group*

$$F \mapsto R\Gamma(B^2 F^\vee, \mathcal{O}).$$

Proof. It suffices to prove that $\mathcal{G}_A \simeq \mathcal{F}_A$, where $\mathcal{F}_A$ is as defined in Proposition 6.1. Using faithfully flat descent along $\mathrm{Spec}\, A \to B^2 F^\vee$, one may check (similar to [MR23, Prop. 6.2.3]) that $R\Gamma(B^2F^\vee, \mathcal{O})$ can be naturally viewed as an object of $\mathcal{G}_A$ and the description in the proposition indeed defines a functor. This also determines a functor $\mathcal{F}_A \to \mathcal{G}_A$. Let $B \in \mathcal{G}_A$ and consider the naturally augmented A-algebra $A \otimes_B A$. Let $V := H^2(B)$. Similar to Proposition 6.1, using the bar complex and the double speed Postnikov filtration on B, one may equip $A \otimes_B A$ with an increasing exhaustive filtration Fil^* such that the graded pieces gr^n are computed by $\mathrm{Sym}^n(V[1])[-2n] \simeq (\wedge^n V)[-n]$. This shows that $B \mapsto A \otimes_B A$ indeed determines a functor from $\mathcal{G}_A \to \mathcal{F}_A$.

As before, since the map $B \to A$ is coconnectively faithfully flat, it follows that $\mathrm{Spec}\, A \to \mathrm{Spec}\, B$ is an effective epimorphism. Writing $\mathrm{Spec}\,(A \otimes_B A) = BG$ by using Proposition 6.1, we see that $\mathrm{Spec}\, B \simeq B^2 G$; in particular the latter is an affine stack. Now we can conclude that the functors described above are inverse to each other. $\square$

Note that as a corollary of the above proof, for a formal group F, one sees that the stack $B^n F$ is affine for $n = 1, 2$. We show that this is true in general.

Proposition 6.4. *Let A be a discrete ring. Let F be an r-dimensional formal group over A. Then $B^n F^\vee$ is an affine stack for all n.*

Proof. We give two proofs. For the first proof, we note that in order to prove that $B^n F^\vee$ is an affine stack, we may argue locally and may assume that F is a formal group law of dimension r. By using [Haz12, Thm. 11.1.5] and [Haz12, Lem. 11.4.7], it is enough to argue in the case of the universal formal group law H_U of dimension m defined over the polynomial ring $\mathbb{Z}[U]$. By Proposition 4.19, it follows that $BH_U^\vee$ is an affine stack. Since $\mathbb{Z}[U]$ is a polynomial algebra (in infinitely many variables), it has the property that every finitely presented A-module has finite flat dimension. Therefore, by Proposition 4.14, we obtain that $B^n H_U^\vee$ is an affine stack, as desired.

Now we give sketch of a second argument, which takes a more direct approach. Let $V := \mathrm{Hom}(F^\vee, \mathbb{G}_a)$. By our hypothesis, it follows V is a projective A-module of rank r. This implies that the natural augmentation map $\Gamma^*(V[-n]) \to A$ is coconnectively faithfully flat. We may suppose that $n \geq 1$. Further, we may choose an isomorphism $F \simeq \mathrm{Sym}^*(V^*)^\wedge$ of algebras, which equivalently produces an isomorphism $\mathcal{O}(F^\vee) \simeq \Gamma^*(V)$ of cocommutative coalgebras. We note the following lemma:

Lemma 6.5. *For $n \geq 1$, we have an isomorphism $R\Gamma(B^n F^\vee, \mathcal{O}) \simeq \Gamma^*(V[-n])$ of E_1-algebras.*

Proof. Note that we have an isomorphism $\mathcal{O}(F^\vee) \simeq \Gamma^*(V)$ of augmented cocommutative coalgebras. In order to compute $R\Gamma(B^nF^\vee, \mathcal{O})$, one may apply faithfully flat descent along $* \to B^nG$; by virtue of the cobar construction, we inductively obtain that $R\Gamma(B^{n-1}F^\vee, \mathcal{O})$ is isomorphic to $\Gamma^*(V[-n+1])$ as augmented coalgebras. Once again, by faithfully flat descent, $R\Gamma(B^nF^\vee, \mathcal{O})$ is obtained by applying the cobar construction to the augmented coalgebra $R\Gamma(B^{n-1}F^\vee, \mathcal{O})$. By [Lur17, Thm. 5.2.2.17], it follows that we have an isomorphism $R\Gamma(B^nF^\vee, \mathcal{O}) \simeq \Gamma^*(V[-n])$ of E_1-algebras. In fact, by using the iterated cobar construction (see [Lur17, § 5.2.3]), by a similar argument, one obtains an isomorphism $R\Gamma(B^nF^\vee, \mathcal{O}) \simeq \Gamma^*(V[-n])$ of E_n-algebras for all $n \geq 0$. □

Since the augmentation map $\Gamma^*(V[-n]) \to A$ is coconnectively faithfully flat, it follows that $R\Gamma(B^nF^\vee, \mathcal{O}) \to A$ is also coconnectively faithfully flat. By Proposition 3.5, it follows that $\operatorname{Spec} A \to \operatorname{Spec} R\Gamma(B^nF^\vee, \mathcal{O})$ is an fpqc epimorphism. Note that for $n \geq 1$, we have a pullback square

$$\begin{tikzcd} \operatorname{Spec}(R\Gamma(B^{n-1}F^\vee, \mathcal{O})) \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec}(R\Gamma(B^nF^\vee, \mathcal{O})). \end{tikzcd}$$

Indeed, to show that the above is a pullback diagram, we need to show that the natural map

$$A \otimes_{R\Gamma(B^nF^\vee, \mathcal{O})} A \to R\Gamma(B^{n-1}F^\vee, \mathcal{O})$$

is an isomorphism of derived rings. To this end, it is enough to show that the natural map is an isomorphism in the derived category of A-modules. However, this follows from the isomorphism $A \otimes_{\Gamma^*(V[-n])} A \simeq \Gamma^*(V[1-n])$ and the isomorphism of E_1-algebras $R\Gamma(B^nF^\vee, \mathcal{O}) \simeq \Gamma^*(V[-n])$ from Lemma 6.5.

Now, by induction, we may assume that $B^{n-1}F^\vee$ is an affine stack, i.e., $B^{n-1}F^\vee \simeq \operatorname{Spec} R\Gamma(B^{n-1}F^\vee, \mathcal{O})$. Since the map $\operatorname{Spec} A \to \operatorname{Spec} R\Gamma(B^nF^\vee, \mathcal{O})$ is an effective fpqc epimorphism, descent along the associated Cech cover and the above pullback square implies that we have a natural isomorphism $B^nF^\vee \simeq \operatorname{Spec} R\Gamma(B^nF^\vee, \mathcal{O})$. This gives the claim. □

Let us now explain how to use Proposition 6.3 to reconstruct the 1-dimensional formal group law over $\Sigma := (\operatorname{Spf} \mathbb{Z}_p)^{\Delta}$ from [Dri21], whose Cartier dual is a flat affine group scheme over Σ that is denoted as G_Σ. In [Dri21], G_Σ is constructed by using the stack $\mathbb{G}_m^{\Delta}$, and crucially uses the delta structure of $\mathbb{G}_m$. We give a different approach using the stack $B\mathbb{G}_m$ and Proposition 6.3. Let $\mu : (B\mathbb{G}_m)^{\Delta} \to \Sigma$ denote the natural map. Then we have $R^*\mu_*\mathcal{O} \simeq \operatorname{Sym}^* R^2\mu_*\mathcal{O}$ as objects of $D_{p-\text{complete}}(\Sigma)$ (cf. [Mon22a]), moreover, $R^2\mu_*\mathcal{O} \simeq \mathcal{O}_\Sigma\{-1\}$. By using Proposition 6.3, it follows that there exits a 1-dimensional formal group F_Σ over Σ such that $\operatorname{Spec} R\mu_*\mathcal{O} \simeq B^2F_\Sigma^\vee$. The next proposition shows that this recovers the construction from [Dri21].

Proposition 6.6. *In the above set up, we have $F_\Sigma^\vee \simeq G_\Sigma$.*

Proof. Our proof will freely use the properties of G_Σ as studied in [Dri21]. More precisely, by using [Dri21, Cor. 2.7.3], it follows that we have a pullback square

$$\begin{tikzcd} B\mathbb{G}_m \arrow[r] \arrow[d] & * \arrow[d] \\ B(\mathbb{G}_m^{\Delta}) \arrow[r] & B^2G_\Sigma. \end{tikzcd}$$

Since $R\Gamma(B\mathbb{G}_m, \mathcal{O}) \simeq k$, it follows that $R\Gamma(B^2G_\Sigma, \mathcal{O}) \simeq R\Gamma(B(\mathbb{G}_m^{\Delta}), \mathcal{O})$. By [Dri21, Thm. 2.7.5], G_Σ is dual to a 1-dimensional formal group over Σ. Let $u : B(\mathbb{G}_m^{\Delta}) \to \Sigma$ denote the natural map. By [Dri21, Thm. 2.7.10] and Proposition 6.3, it follows that $R^2u_*\mathcal{O} \simeq \mathcal{O}_\Sigma\{-1\}$ and

$R^*u_*\mathcal{O} \simeq \mathrm{Sym}^* R^2u_*\mathcal{O}$. Now, using the discussion before Proposition 6.6, we see that the natural map $B(\mathbb{G}_m^{\mathbb{\Delta}}) \to (B\mathbb{G}_m)^{\mathbb{\Delta}}$ induces an isomorphism $\operatorname{Spec} R\mu_*\mathcal{O} \simeq \operatorname{Spec} Ru_*\mathcal{O}$. This gives a natural map $B^2F_\Sigma^\vee \to B^2G_\Sigma$ of pointed stacks, which by construction and the affineness of B^2G_Σ (Proposition 6.4), is an isomorphism. This gives $F_\Sigma^\vee \simeq G_\Sigma$, as desired. ☐

Below, we will use our approach of reconstructing G_Σ using $B\mathbb{G}_m$ to construct certain refinements, which we will denote by $G_{\Sigma'}$ and $G_{\Sigma''}$. To this end, we will need the following lemma, which computes the full Nygaard filtration on absolute prismatic cohomology of $B\mathbb{G}_m$.

Lemma 6.7. *Let R be a quasiregular semiperfectoid algebra. We have a natural isomorphism*

$$\bigoplus_{i\geq 0} \mathrm{Fil}^{m-i}_{\mathrm{Nyg}} \mathbb{\Delta}_R \{n-i\}[-2i] \xrightarrow{\sim} \mathrm{Fil}^m_{\mathrm{Nyg}} R\Gamma_{\mathbb{\Delta}}(B\mathbb{G}_m)\{n\}.$$

Proof. The proof is similar to [BL22a, Lem. 9.1.4]. Let t denote the tautological class in $H^2_{\mathrm{syn}}(B\mathbb{G}_m, \mathbb{Z}_p(1))$ corresponding to the identity map $B\mathbb{G}_m \to B\mathbb{G}_m$. The classes t^i for $i \in \mathbb{N}$ induce a natural map

$$\bigoplus_{i\geq 0} \mathrm{Fil}^{m-i}_{\mathrm{Nyg}} \mathbb{\Delta}_R \{n-i\}[-2i] \to \mathrm{Fil}^m_{\mathrm{Nyg}} R\Gamma_{\mathbb{\Delta}}(B\mathbb{G}_m)\{n\}.$$

We will show that this is an isomorphism. For $m \leq 0$, the claim follows from the discussion before Proposition 6.6. By induction, it would be enough to show that the induced map

$$\bigoplus_{i\geq 0} \mathrm{gr}^{m-i}_{\mathrm{Nyg}} \mathbb{\Delta}_R \{n-i\}[-2i] \to \mathrm{gr}^m_{\mathrm{Nyg}} R\Gamma_{\mathbb{\Delta}}(B\mathbb{G}_m)\{n\}$$

is an isomorphism. Let S be a perfectoid algebra that maps surjectively to R. Using this, the above map can be identified with the induced map

$$\bigoplus_{i\geq 0} \mathrm{Fil}^{m-i}_{\mathrm{conj}} \overline{\mathbb{\Delta}}_R \{m-i\}[-2i] \to \mathrm{Fil}^m_{\mathrm{conj}} R\Gamma_{\mathrm{HT}}(B\mathbb{G}_m)\{m\}.$$

To check that the above map is an isomorphism, we can again pass to graded pieces. Then $\mathrm{gr}^m_{\mathrm{conj}} R\Gamma_{\mathrm{HT}}(B\mathbb{G}_m)\{m\}$ is naturally isomorphic to $\wedge^m \mathbb{L}_{B\mathbb{G}_m/S}[-m]^{\wedge p}$. By the transitivity fiber sequence for cotangent complex, $\mathbb{L}^{\wedge p}_{B\mathbb{G}_m/S} \simeq \mathcal{O}[-1] \oplus \mathbb{L}^{\wedge p}_{R/S}$. Therefore, we have $\wedge^m \mathbb{L}_{B\mathbb{G}_m/S}[-m]^{\wedge p} \simeq \bigoplus_{i\geq 0} \wedge^{m-i} \mathbb{L}^{\wedge p}_{R/S}[-i-m] \simeq \mathrm{gr}^{m-i}_{\mathrm{conj}} \overline{\mathbb{\Delta}}_R \{m-i\}[-2i]$. This yields the required isomorphism on graded pieces, which finishes the proof. ☐

Now let $\Sigma' := (\operatorname{Spf} \mathbb{Z}_p)^{\mathcal{N}}$. There is a natural map $u : (B\mathbb{G}_m)^{\mathcal{N}} \to \Sigma'$. In this language, Lemma 6.7 admits the following translation:

Proposition 6.8. *In the above set up, $R^*u_*\mathcal{O} \simeq \mathrm{Sym}^* R^2u_*\mathcal{O}$; moreover, $R^2u_*\mathcal{O} \simeq \mathcal{O}_{\Sigma'}\{-1\}$.*

Proof. Let R be a quasiregular semiperfectoid algebra. By a result proven by Bhatt–Lurie (see [Bha23, Cor. 5.5.11]), we have a natural isomorphism of stacks

$$(\operatorname{Spf} R)^{\mathcal{N}} \simeq \left(\operatorname{Spf} \bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i_{\mathrm{Nyg}} \mathbb{\Delta}_R \{i\}\right)/\mathbb{G}_m.$$

Under this isomorphism, the line bundle $\mathcal{O}_{(\operatorname{Spf} R)^{\mathcal{N}}}\{-1\}$ corresponds to the graded $\bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i_{\mathrm{Nyg}} \mathbb{\Delta}_R\{i\}$-module $\bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^{i-1}_{\mathrm{Nyg}} \mathbb{\Delta}_R \{i-1\}$. Now let $v\colon (B\mathbb{G}_{m,R})^{\mathcal{N}} \to (\operatorname{Spf} R)^{\mathcal{N}}$ denote the natural map. As a consequence of the previous discussion and Lemma 6.7, we see that $R^2v_*\mathcal{O} \simeq \mathcal{O}_{(\operatorname{Spf} R)^{\mathcal{N}}}\{-1\}$ and $R^*v_*\mathcal{O} \simeq \mathrm{Sym}^* R^2v_*\mathcal{O}$. By quasisyntomic descent, we obtain the desired claim. ☐

Proposition 6.9. *In the above set up, there exists a 1-dimensional formal group over Σ' denoted by $F_{\Sigma'}$ (that extends F_Σ) such that $\operatorname{Spec} Ru_*\mathcal{O} \simeq B^2F^\vee_{\Sigma'}$.*

Proof. Follows from Proposition 6.3. □

Remark 6.10. Let Σ'' be as defined in [Dri20], which is also denoted by $(\mathrm{Spf}\,\mathbb{Z}_p)^{\mathrm{syn}}$. Using the coequalizer description of Σ'' from [Dri20, 8.2.1], it follows that $F_{\Sigma'}$ from Proposition 6.9 glues to a 1-dimensional formal group $F_{\Sigma''}$ over Σ''.

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