

Complex Numbers

Write $z \in \mathbb{C}$ for:

$$z = a + ib, \quad i^2 = -1. \quad (a, b \in \mathbb{R})$$

Note: $z_1 = z_2$ if $\begin{matrix} z_1 = a_1 + ib_1 \\ z_2 = a_2 + ib_2 \end{matrix}$

$$\text{with } \begin{matrix} a_1 = a_2 \\ b_1 = b_2 \end{matrix}.$$

Notation: $\begin{matrix} \operatorname{Re}(z) = a \\ \operatorname{Im}(z) = b \end{matrix}.$

- Addition

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

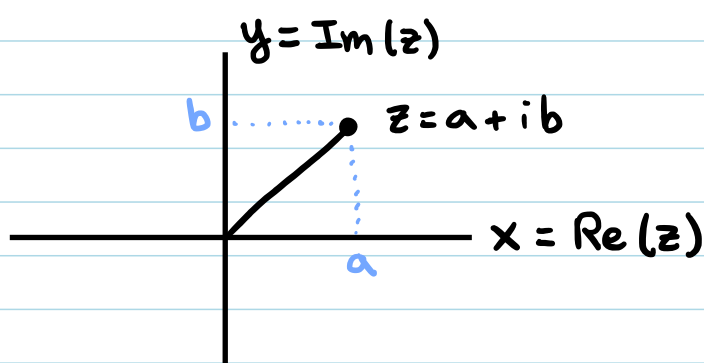
- Multiplication

$$\begin{aligned} (a + ib)(c + id) &= ac + iad + ibc + i^2 bd \\ &= (ac - bd) + i(ad + bc) \end{aligned}$$

- Division

$$\begin{aligned} \frac{a + ib}{c + id} &= \frac{a + ib}{c + id} \cdot \frac{c - id}{c - id} \\ &= \frac{ac + bd}{c^2 + d^2} + i \left(\frac{bc - ad}{c^2 + d^2} \right). \end{aligned}$$

Cartesian Coordinate Representation



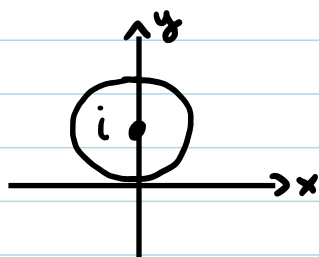
Think of:
 $z = a + ib \in \mathbb{C}$
as a vector
 $(a, b) \in \mathbb{R}^2.$

Modulus: $|z| = (a^2 + b^2)^{\frac{1}{2}}$.

Note: this is the length of the vector (a, b) .

Note: $|z_1 - z_2| = ((a_1 - a_2)^2 + (b_1 - b_2)^2)^{\frac{1}{2}}$, $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$
is the distance between z_1 and z_2 .

ex) Sketch $|z - i| = 1$.



Circle of radius 1
centred at i .

ex) Describe the set $|z| = 3|z - 1|$.

$$|z|^2 = 9|z - 1|^2 \quad \text{Write } z = x + iy.$$
$$x^2 + y^2 = 9((x - 1)^2 + y^2)$$

$$\Rightarrow \frac{x^2 + y^2}{9} = x^2 - 2x + 1 + y^2$$

$$\Rightarrow \frac{8}{9}x^2 + \frac{8}{9}y^2 - 2x + 1 = 0$$

$$\Rightarrow x^2 + y^2 - \frac{9}{4}x + \frac{9}{8} = 0$$

$$\Rightarrow \left(x - \frac{9}{8}\right)^2 - \frac{9^2}{8^2} + y^2 + \frac{9}{8} = 0$$

$$\Rightarrow \left(x - \frac{9}{8}\right)^2 + y^2 = \frac{9}{8^2}$$

$\therefore$ Circle of radius $\frac{3}{8}$ centred at $z = \frac{9}{8}$.

More identities:

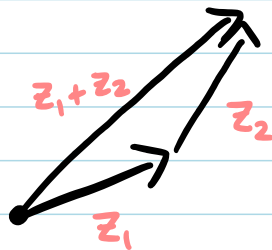
$$|z_1 z_2| = |z_1| |z_2| \quad \text{Derivation omitted}$$

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$

Triangle Inequality

(same proof as in multivariable calculus)

$$|z_1 + z_2| \leq |z_1| + |z_2|$$



$$z_1 = a_1 + i b_1$$

$$z_2 = a_2 + i b_2$$

$$z_1 + z_2 \in \mathbb{C} \quad \rightsquigarrow \quad (a_1, b_1) + (a_2, b_2) \in \mathbb{R}^2$$

Reverse Triangle Inequality

$$||z_1| - |z_2|| \leq |z_1 - z_2|$$

$$\text{Proof: } |z_1| = |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2| \quad \text{by triangle ineq}$$

$$\therefore |z_1| - |z_2| \leq |z_1 - z_2|$$

$$\text{Redo starting from } |z_2|, \text{ get } |z_2| - |z_1| \leq |z_1 - z_2|$$

$$\Rightarrow ||z_1| - |z_2|| \leq |z_1 - z_2|$$

ex) Find smallest $M > 0$ s.t. $\frac{1}{|z-6|} \leq M$ for all $|z+1| \leq 1$.

$$\text{Need: } |z-6| \geq M^{-1}$$

$$\text{Reverse triangle: } |z-6| = |z+1-7| \geq ||z+1| - 7|$$

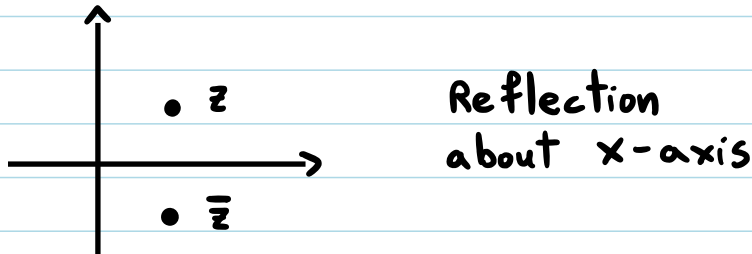
$$= 7 - |z+1| \quad \text{since } |z+1| \leq 1$$

$$\geq 7 - 1 \quad \text{since } -|z+1| \geq -1$$

$$\therefore |z-6| \geq 6 \Rightarrow M = 1/6. \quad \text{Note: } z=0 \text{ attains this bound.}$$

Complex conjugate:

$$z = a + ib \quad \rightsquigarrow \quad \bar{z} = a - ib.$$



Properties of complex conjugate

$$(a) \quad \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

$$(b) \quad \overline{\bar{z}} = z$$

$$(c) \quad \operatorname{Re}(z) = \frac{1}{2}(z + \bar{z}), \quad \operatorname{Im}(z) = \frac{1}{2i}(z - \bar{z}).$$

Proof: $z = x + iy, \quad \bar{z} = x - iy$
 $z + \bar{z} = 2x$
 $z - \bar{z} = 2iy.$

$$(d) \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

$$(e) \quad |z|^2 = z \bar{z} = x^2 + y^2$$

$$(f) \quad |z| = |\bar{z}|$$

Some more remarks :

$$(i) \quad \begin{aligned} \operatorname{Re}(z) &\leq |z| & x &\leq |x| \leq (x^2 + y^2)^{\frac{1}{2}} = |z| \\ \operatorname{Im}(z) &\leq |z| & y &\leq |y| \leq (x^2 + y^2)^{\frac{1}{2}} = |z| \end{aligned}$$

$$(ii) \quad \operatorname{Re}(iz) = -\operatorname{Im}(z)$$

$$(iii) \quad \operatorname{Re}\left(\frac{1}{z}\right) > 0 \iff \operatorname{Re}(z) > 0$$
$$\operatorname{Re}\left(\frac{1}{z}\right) = \operatorname{Re}\left(\frac{\bar{z}}{z\bar{z}}\right) = \frac{1}{|z|^2} \operatorname{Re}(\bar{z}) = \frac{1}{|z|^2} \operatorname{Re}(z).$$

Euler's Formula : $e^{i\varphi} = \cos \varphi + i \sin \varphi$.

Motivation: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$$e^{i\varphi} = 1 + i\varphi + \frac{(i\varphi)^2}{2!} + \frac{(i\varphi)^3}{3!} + \frac{(i\varphi)^4}{4!} + \dots$$

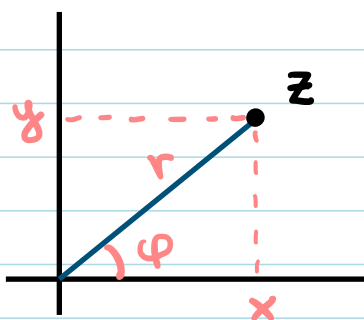
$$= \underbrace{\left(1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \dots\right)}_{\cos \varphi} + i \underbrace{\left(\varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \dots\right)}_{\sin \varphi}$$

ex) $i = e^{i\pi/2}$

ex) $-1 = e^{i\pi}$

ex) $-i = e^{-i\pi/2}$

Polar Coordinates



$$z = x + iy$$

$$x = r \cos \varphi$$

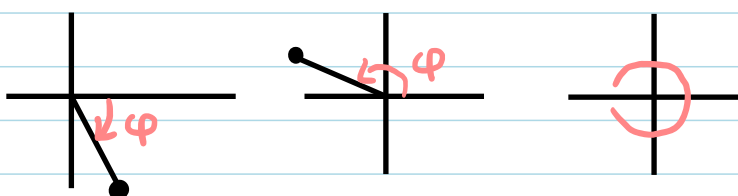
$$z = r e^{i\varphi}$$

$$y = r \sin \varphi$$

$$\begin{aligned} z &= r e^{i\varphi} = r (\cos \varphi + i \sin \varphi) \\ &= \underbrace{(r \cos \varphi)}_x + i \underbrace{(r \sin \varphi)}_y \end{aligned}$$

$$|z| = \left(r^2 \cos^2 \varphi + r^2 \sin^2 \varphi \right)^{\frac{1}{2}} = r.$$

Notation: $\text{Arg}(z) = \varphi$ for $\varphi \in (-\pi, \pi]$.

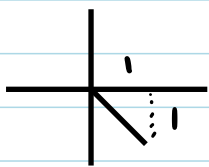


Sometimes use lower case for multivalued argument function:

$$\arg(z) = \varphi + 2\pi K, \quad K = 0, \pm 1, \pm 2, \dots$$
$$\varphi \in (-\pi, \pi].$$

ex) Write $1-i$ in polar form, and compute $\text{Arg}(1-i)$.

$$|1-i| = \sqrt{1+1} = \sqrt{2}$$



$$\cos \varphi = 1/\sqrt{2}$$

$$\sin \varphi = -1/\sqrt{2}$$

$$(1-i) = \sqrt{2} e^{-i\pi/4}$$

$$\text{Arg}(1-i) = -\frac{\pi}{4}.$$

Some properties:

$$(a) e^{-i\varphi} = \frac{1}{e^{i\varphi}} = \overline{e^{i\varphi}}$$

$$\arg \bar{z} = \arg \frac{1}{z}$$

$$(b) |e^{i\varphi}| = 1$$

$$(c) e^{i\varphi_1} e^{i\varphi_2} = e^{i(\varphi_1 + \varphi_2)}$$

$$(d) e^{i\varphi_1} / e^{i\varphi_2} = e^{i(\varphi_1 - \varphi_2)}$$

$$(e) e^{i(\varphi + 2\pi K)} = e^{i\varphi}, \quad K = 0, \pm 1, \pm 2, \dots$$

Proofs: omitted. Expand $e^{i\varphi} = \cos \varphi + i \sin \varphi$.

Note: complex multiplication in polar form:

$$z_1 z_2 = (r_1 e^{i\varphi_1}) (r_2 e^{i\varphi_2}) \\ = (r_1 r_2) e^{i(\varphi_1 + \varphi_2)}.$$

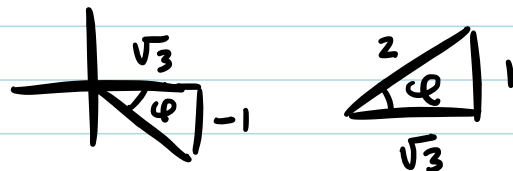
Note: complex conjugation:

$$z = r e^{i\varphi} \Rightarrow \bar{z} = r e^{-i\varphi}.$$

ex) Write $\frac{1+i}{\sqrt{3}-i}$ in polar form.

$$1+i = \sqrt{2} e^{i\pi/4}$$

$$\sqrt{3}-i = 2 e^{-i\pi/6}$$



$$\Rightarrow \frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}}{2} \frac{e^{i\pi/4}}{e^{-i\pi/6}} = \frac{1}{\sqrt{2}} e^{i\frac{5\pi}{12}}.$$

De Moivre's Formula

$$(e^{i\varphi})^n = e^{in\varphi}.$$

$$\Rightarrow (\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi.$$

ex) Write $\sin^3\varphi$ in terms of $\sin\varphi$ and $\sin 3\varphi$.

$$(\cos\varphi + i\sin\varphi)^3 = \cos 3\varphi + i\sin 3\varphi.$$

(*)

$$\cos^3\varphi + 3i\cos^2\varphi\sin\varphi + 3i^2\cos\varphi\sin^2\varphi + i^3\sin^3\varphi.$$

Take $\text{Im}(\ast)$:

$$3 \cos^2 \varphi \sin \varphi - \sin^3 \varphi = \sin 3 \varphi.$$

$$3(1 - \sin^2 \varphi) \sin \varphi - \sin^3 \varphi = \sin 3 \varphi.$$

$$3 \sin \varphi - 4 \sin^3 \varphi = \sin 3 \varphi.$$

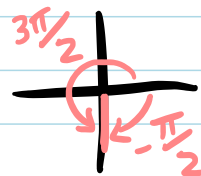
$$\therefore \sin^3 \varphi = \frac{3}{4} \sin \varphi - \frac{1}{4} \sin 3 \varphi.$$

Note: $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

$$\text{Arg}(z_1 z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2).$$

e.g. $z_1 = e^{3\pi i/4} = z_2$

$$z_1 z_2 = e^{3\pi i/2}.$$



$$\text{Arg}(z_1 z_2) = -\frac{\pi}{2}$$

$$\text{Arg}(z_1) + \text{Arg}(z_2) = \frac{3\pi}{4} + \frac{3\pi}{4}.$$

Note: $e^{i\varphi_1} = e^{i\varphi_2}$

$$\Leftrightarrow \varphi_1 = \varphi_2 + 2\pi k, \quad k = 0, \pm 1, \pm 2, \dots$$

If $\varphi_1 = \varphi_2 + 2\pi k$, then $e^{i\varphi_1} = e^{i\varphi_2} \underbrace{e^{i2\pi k}}_{=1}.$

If $e^{i\varphi_1} = e^{i\varphi_2}$, then $e^{i(\varphi_1 - \varphi_2)} = 1.$

$$\Rightarrow \cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2) = 1 + 0i.$$

$$\Rightarrow \varphi_1 - \varphi_2 = 2\pi k, \quad k = 0, \pm 1, \pm 2, \dots$$

Complex exponential function

$$e^z := e^x e^{iy}, \quad z = x + iy.$$

$$e^z = e^x (\cos y + i \sin y).$$

Sometimes write: $\exp(z)$ instead of e^z .

Properties:

$$(i) \quad e^{z_1 + z_2} = e^{z_1} e^{z_2}$$

$$(ii) \quad e^{z + 2\pi i} = e^z$$

$$(iii) \quad e^{z + \pi i} = -e^z$$

$$(iv) \quad \overline{e^z} = e^{\bar{z}}$$

$$(v) \quad e^z = 1 \quad \Leftrightarrow \quad z = i(2\pi k), \quad k = 0, \pm 1, \pm 2, \dots$$

$$(vi) \quad |e^z| \leq 1 \quad \text{if} \quad \operatorname{Re}(z) \leq 0$$

$$|e^z| = |e^x| |e^{iy}| = e^x \leq 1 \quad \text{if} \quad x \leq 0.$$

Often used fact:

$$e^{z_1} = e^{z_2} \quad \Leftrightarrow \quad z_1 = z_2 + i(2\pi k).$$

$$k = 0, \pm 1, \pm 2, \dots$$

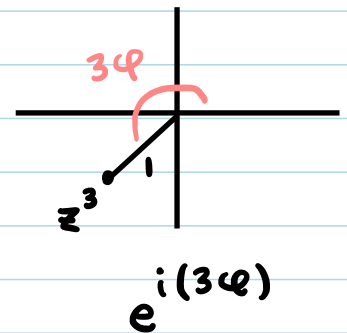
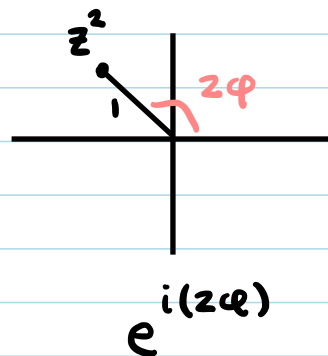
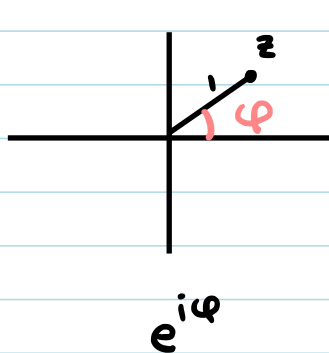
$$e^{z_1 - z_2} = 1 \quad \Leftrightarrow \quad e^{x_1 - x_2} e^{i(y_1 - y_2)} = 1$$

$$\Leftrightarrow e^{x_1 - x_2} \cos(y_1 - y_2) + i e^{x_1 - x_2} \sin(y_1 - y_2) = 1$$

$$\Leftrightarrow \begin{cases} \sin(y_1 - y_2) = 0 \\ e^{x_1 - x_2} \cos(y_1 - y_2) = 1 \end{cases} \quad \Leftrightarrow \begin{cases} y_1 - y_2 = \pi k \\ e^{x_1 - x_2} \cos(\pi k) = 1 \end{cases}$$

$$\Leftrightarrow x_1 = x_2, \quad y_1 - y_2 = 2\pi k$$

Roots of complex numbers



ex) $1^2 = 1$
 $(-1)^2 = 1$

ex) $1^4 = 1$
 $i^4 = 1$
 $(-1)^4 = 1$
 $(-i)^4 = 1$

ex) $1^3 = 1$
 $\left(e^{i(2\pi/3)}\right)^3 = 1$
 $\left(e^{i(4\pi/3)}\right)^3 = 1$

Roots of unity: $\omega_n := e^{2\pi i/n}$

The n^{th} roots of unity are:

$$\{1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1}\}.$$

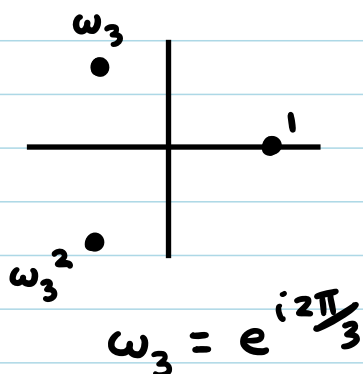
Note: $(\omega_n^k)^n = 1$, $k = 0, \pm 1, \pm 2, \dots$

Note: $\omega_n^n = 1$, $\omega_n^{n+1} = \omega_n$, $\omega_n^{n+2} = \omega_n^2$, $\dots$

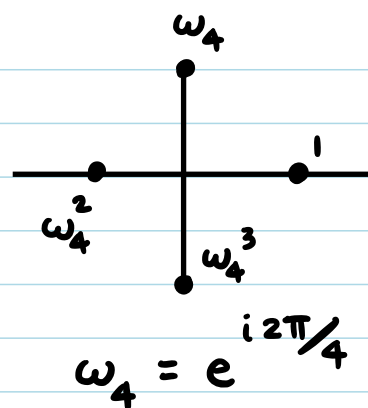
$$\Rightarrow \{\omega_n^k\}_{k \in \mathbb{Z}} = \{1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1}\}$$

there are n n -th roots of unity

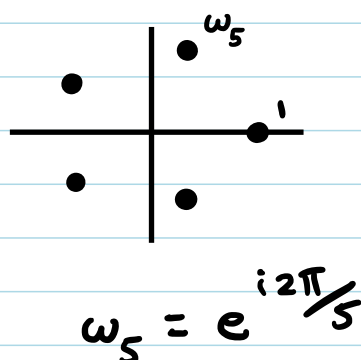
ex)



ex)



ex)



Remark: The solutions of $z^n = 1$ are:

$$z = \omega_n^k, \quad k = 0, 1, \dots, n-1.$$

$$z = \exp\left(i \frac{2\pi k}{n}\right), \quad k = 0, 1, \dots, n-1$$

There are n solutions to $z^n = 1$.

$$\begin{aligned} z = re^{i\theta}. \text{ Eqn } z^n = 1 &\Leftrightarrow e^{n \ln r + i n \theta} = e^0 \\ &\Leftrightarrow n \ln r + i n \theta = i (2\pi k) \\ &\Leftrightarrow r = 1 \text{ and } \theta = \frac{2\pi k}{n}. \end{aligned}$$

ex) Show $\omega_n + \omega_n^2 + \dots + \omega_n^{n-1} = -1$.

start from geometric series

$$1 + \omega + \omega^2 + \dots + \omega^{n-1} = \frac{1 - \omega^n}{1 - \omega}.$$

$$\Rightarrow 1 + \omega_n + \omega_n^2 + \dots + \omega_n^{n-1} = 0.$$

How to solve $z^n = \zeta$

① Write $\zeta = \rho e^{i\theta}$.

② Set $z = r e^{i\varphi}$

③ Eqn is: $r^n e^{i(n\varphi)} = \rho e^{i\theta}$

④ Soln are: $r = \rho^{1/n}$

$$i n \varphi = i \theta + 2\pi k i, \quad k = 0, \pm 1, \pm 2, \dots$$

⑤ Write using roots of unity:

$$z = \rho^{1/n} \exp\left(\frac{i\theta}{n} + i\left(\frac{2\pi k}{n}\right)\right) \quad k = 0, \dots, n-1$$

$$z = \rho^{1/n} e^{i\theta/n} \omega_n^k, \quad k = 0, \dots, n-1.$$

ex) Solve $z^5 = 1 + i$.

$$1 + i = \sqrt{2} e^{i\pi/4}$$

$$z^5 = \sqrt{2} e^{i\pi/4}$$

$$z = (\sqrt{2})^{1/5} e^{i\pi/20} \omega_5^k, \quad k = 0, 1, 2, 3, 4. \quad \omega_5 = e^{2\pi i/5}$$

ex) Solve $z^3 = (z+1)^3$.

$$\left(\frac{z+1}{z}\right)^3 = 1 \Rightarrow 1 + z^{-1} = \omega_3^k, \quad k = 0, 1, 2.$$

$$\Rightarrow (1 - \omega_3^k) z = -1$$

$$\Rightarrow z = \frac{-1}{1 - \omega_3^k}, \quad k = 1, 2. \quad \text{Throw out } k = 0.$$