

Residue Theorem

Let $f(z)$ be analytic on $0 < |z - z_0| < r$.

Laurent expansion at z_0 :

$$f(z) = \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Def: $\text{Res}(f, z_0) = a_{-1}$. coefficient of $(z - z_0)^{-1}$.

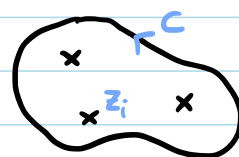
Last Time: if $f = \frac{P}{Q}$ with $P(z_0) \neq 0$, $P(z)$ and $Q(z)$ analytic near z_0 .
 $Q(z_0) = 0$
 $Q'(z_0) \neq 0$

$$\Rightarrow \text{Res}(f, z_0) = \frac{P(z_0)}{Q'(z_0)}.$$

Thm: (Residue Theorem)

Let $f(z)$ be analytic on and inside a CCW loop C except at singular points $z_1, \dots, z_n$ inside C . Then:

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f, z_k).$$



Proof: By Green's Thm

$$\int_C f(z) dz = \sum_{k=1}^n \int_{C_k} f(z) dz, \quad C_k = |z - z_k| = \varepsilon \text{ CCW}.$$

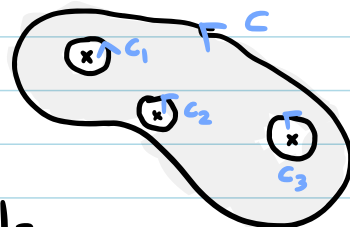
Focus near a given z_k :

$$\int_{C_k} f dz = \int_{C_k} \left(\sum_{n=1}^{\infty} a_{-n} (z - z_k)^{-n} + \sum_{n=0}^{\infty} a_n (z - z_k)^n \right) dz$$

$$= 2\pi i a_{-1}, \quad \text{since: } \int_{C_k} (z - z_k) dz = \begin{cases} 0 & k \neq -1 \\ 2\pi i & k = -1 \end{cases}$$

$$\Rightarrow \int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f, z_k).$$

□



ex) Calculate $\int_C z^3 e^{\frac{1}{z}} dz$, $C = \{|z|=1\}$ ccw.

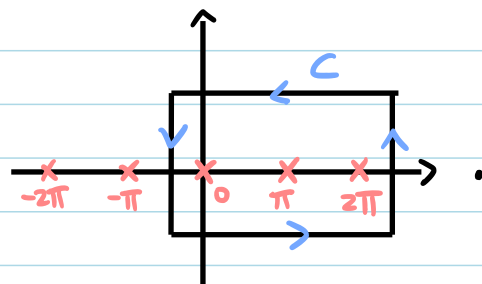
Find Laurent series at 0:

$$\begin{aligned} z^3 e^{\frac{1}{z}} &= z^3 \left(1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \frac{z^{-4}}{4!} + \dots \right) \\ &= z^3 + z^2 + \frac{z}{2!} + \frac{1}{3!} + \frac{1}{4!z} + \dots \end{aligned}$$

$$a_{-1} = \frac{1}{4!}$$

$$\Rightarrow \int_C z^3 e^{\frac{1}{z}} dz = (2\pi i) \left(\frac{1}{4!} \right) = \frac{\pi i}{12}$$

ex) Calculate $\int_C \frac{1}{\sin z} dz$, where:



$$\int_C f dz = 2\pi i \left(\text{Res}(f, 0) + \text{Res}(f, \pi) + \text{Res}(f, 2\pi) \right)$$

$$\begin{aligned} f(z) &= \frac{1}{\sin z} = \frac{P(z)}{Q(z)}, \quad P=1, \quad Q'=\cos z \\ \Rightarrow \text{Res}(f, 0) &= \frac{1}{Q'(0)} = 1 \\ \Rightarrow \text{Res}(f, \pi) &= \frac{1}{Q'(\pi)} = -1 \\ \Rightarrow \text{Res}(f, 2\pi) &= \frac{1}{Q'(2\pi)} = 1 \end{aligned}$$

$$\Rightarrow \int_C f dz = 2\pi i (1 - 1 + 1) = 2\pi i$$

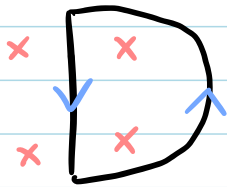
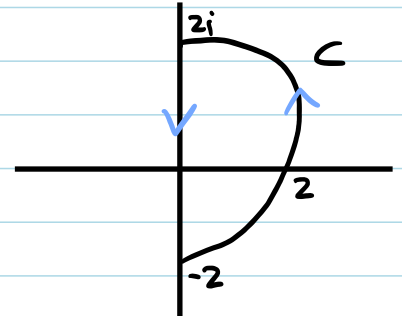
ex) Calculate $\int_C \frac{1}{1+z^4} dz$, where C is a semi-circle of radius 2 on the right-half plane.

Solve: $1+z^4=0$.

$z^4 = -1$
 $z = e^{i\pi/4} \omega_4^k, \quad k=0,1,2,3.$

$\omega_4 = e^{2\pi i/4} = e^{i\pi/2}$

$\Rightarrow z = e^{i\pi/4}, e^{i\pi/4} e^{i\pi/2}, e^{i\pi/4} e^{i\pi}, e^{i\pi/4} e^{i3\pi/2}$
 (Arrows point to the first and fourth roots, labeled "inside C")



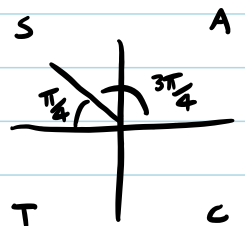
$\int_C f dz = 2\pi i \left(\text{Res}(f, e^{i\pi/4}) + \text{Res}(f, e^{i\pi/4} e^{i3\pi/2}) \right), \quad f = (1+z^4)^{-1}$

Write $f = \frac{P}{Q}$, $P=1$
 $Q' = 4z^3$

$\text{Res}(f, e^{i\pi/4}) = \frac{1}{Q'(e^{i\pi/4})} = \frac{1}{4} e^{-3\pi i/4}$

$\text{Res}(f, e^{i\pi/4} e^{i3\pi/2}) = \frac{1}{Q'(e^{i7\pi/4})} = \frac{1}{4} e^{-21\pi i/4} = \frac{1}{4} e^{-5\pi i/4} = \frac{1}{4} e^{3\pi i/4}$

$\Rightarrow \int_C f dz = \frac{2\pi i}{4} \left(e^{-3\pi i/4} + e^{3\pi i/4} \right) = \pi i \cos\left(\frac{3\pi}{4}\right)$
 $= -\frac{\pi i}{\sqrt{2}}$



$$\text{ex) } \int_{\substack{|z|=1 \\ \text{CCW}}} \frac{dz}{z^2 \sin z} = ?$$

Zeros of $z^2 \sin z$ at $z=0, \pm\pi, \pm2\pi, \dots$ Only $z=0$ in the circle

$$\begin{aligned} \text{Residue at } 0: \frac{1}{z^2 \sin z} &= \frac{1}{z^2 \left(z - \frac{z^3}{3!} + \dots \right)} \\ &= z^{-3} \left(1 - \frac{z^2}{3!} + \dots \right)^{-1} \\ &= z^{-3} \left(1 + \frac{z^2}{3!} + \dots \right) \\ &= z^{-3} + \frac{1}{6} z^{-1} + \dots \Rightarrow \text{Res}(\cdot, 0) = 1/6. \end{aligned}$$

$$\Rightarrow \int = 2\pi i \left(\frac{1}{6} \right) = i \frac{\pi}{3}.$$

Trig Integrals

$$\text{Let } z = e^{i\varphi}. \text{ Then: } \cos \varphi = \frac{1}{2} (z + z^{-1})$$

$$\sin \varphi = \frac{1}{2i} (z - z^{-1})$$

$$d\varphi = \frac{dz}{iz}$$

$$\text{ex) Calculate } \int_0^{2\pi} \frac{d\varphi}{2 + \sin \varphi}.$$

$$\text{Let } z = e^{i\varphi}, dz = i e^{i\varphi} d\varphi$$

$$2 + \sin \varphi = 2 + \frac{1}{2i} (z - z^{-1}) = \frac{1}{2} (4 - iz + iz^{-1}).$$

Integral becomes

$$\int_{|z|=1} \frac{2}{(4 - iz + iz^{-1}) iz} dz$$

$$= \int_{|z|=1} \frac{2dz}{z^2 + 4iz - 1}$$

zeros of denom: $z = \frac{-4i \pm \sqrt{-16+4}}{2} = -2i \pm \sqrt{-3}$
 $= i(-2 \pm \sqrt{3})$.

Only $i(-2 + \sqrt{3})$ inside $|z| \leq 1$.

$$\int_{|z|=1} = 2\pi i \operatorname{Res}(\cdot, \underbrace{i(-2 + \sqrt{3})}_{z_0})$$

$$= (2\pi i) \frac{P(z_0)}{Q'(z_0)}, \quad \begin{aligned} P &= 2 \\ Q &= z^2 + 4iz - 1 \\ Q' &= 2z + 4i \end{aligned}$$

$$= \frac{(2\pi i)(2)}{2i(-2 + \sqrt{3}) + 4i}$$

$$= \frac{2\pi}{\sqrt{3}}$$

ex) Compute $\int_0^{2\pi} \frac{d\varphi}{1 - 2a \cos \varphi + a^2}$ for $0 < a < 1$.

Let $z = e^{i\varphi}$, $dz = iz d\varphi$.

$$1 - 2a \cos \varphi + a^2 = 1 + a^2 - a(z + z^{-1}).$$

$$\begin{aligned} \text{Integral} &= \int_{|z|=1} \frac{dz}{(1 + a^2 - az - az^{-1})iz} \\ &= \int_{|z|=1} \frac{dz}{i(-az^2 + (1+a^2)z - a)} \end{aligned}$$

zeros of denom: $z = \frac{-1-a^2 \pm \sqrt{(1+a^2)^2 - 4(-a)(-a)}}{-2a} = \frac{1+a^2 \pm \sqrt{1-2a^2+a^4}}{2a}$

$$\text{Roots: } z = \frac{1+a^2 \pm \sqrt{(1-a^2)}}{2a} = \frac{1+a^2 \pm (1-a^2)}{2a}$$

$$= 1/a \text{ or } a.$$

Only singularity in $|z| \leq 1$ is $z=a$. ($0 < a < 1$).

$$\text{Integral} = 2\pi i \text{ Res}(\cdot, a)$$

$$= 2\pi i \frac{P(a)}{Q'(a)},$$

$$P=1$$

$$Q = i(-az^2 + (1+a^2)z - a)$$

$$Q' = -2ia z + i(1+a^2)$$

$$= \frac{2\pi i}{-2ia^2 + i(1+a^2)}$$

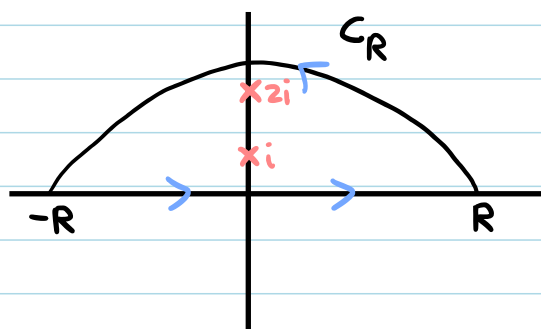
$$= \frac{2\pi}{1-a^2}.$$

Integrals over $(-\infty, \infty)$

ex) Compute $\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)} dx$.

Close the loop:

$$\int_{-R}^R \frac{x^2}{(1+x^2)(4+x^2)} dx + \int_{C_R} \frac{z^2}{(1+z^2)(4+z^2)} dz = 2\pi i \sum \text{Res}$$



$$\text{poles at: } (1+z^2)(4+z^2) = 0$$

$$z = \pm i, \pm 2i.$$

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)} dx + \lim_{R \rightarrow \infty} \underbrace{\int_{C_R} \frac{z^2}{(1+z^2)(4+z^2)} dz}_{\text{this goes to zero}} = 2\pi i \left(\text{Res}(\cdot, i) + \text{Res}(\cdot, 2i) \right)$$

$$\left| \int_{C_R} \frac{z^2}{(1+z^2)(4+z^2)} dz \right| \leq \left(\max_{|z|=R} \frac{|z|^2}{|1+z^2||4+z^2|} \right) \text{Length}(C_R)$$

$$\leq \frac{R^2}{(R^2-1)(R^2-4)} \pi R \quad |1+z^2| \geq |1-|z|^2|$$

$\rightarrow 0$ as $R \rightarrow \infty$.

$$\therefore \int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)} dx = 2\pi i \left(\text{Res}(\cdot, i) + \text{Res}(\cdot, 2i) \right)$$

$$= 2\pi i \left(\frac{P(i)}{Q'(i)} + \frac{P(2i)}{Q'(2i)} \right)$$

where $P(z) = z^2$

$$Q(z) = (1+z^2)(4+z^2)$$

$$Q'(z) = 2z(4+z^2) + (1+z^2)(2z)$$

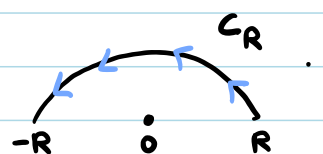
$$= 2\pi i \left(\frac{i^2}{2i(4-1)+0} + \frac{4i^2}{0+(1-4)(4i)} \right)$$

$$= 2\pi \left(-\frac{1}{6} + \frac{1}{3} \right)$$

$$= \frac{\pi}{3}.$$

Lemma: Suppose $|f(z)| \leq \frac{M}{|z|^{1+a}}$, for $M > 0$, $a > 0$, once $|z| \gg 1$.

Then: $\lim_{R \rightarrow \infty} \int_{C_R} f dz = 0$, where:



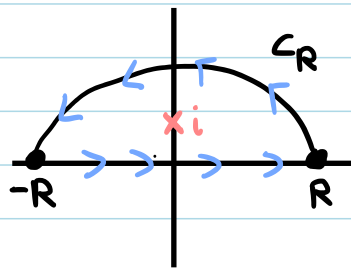
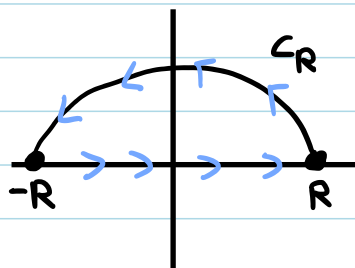
Proof of lemma: $\left| \int_{C_R} f dz \right| \leq \max_{C_R} |f| \cdot \text{Length}(C_R)$

$$\leq \frac{M}{R^{1+a}} \pi R$$

$$\rightarrow 0 \text{ as } R \rightarrow \infty. \quad \square$$

ex) Calculate $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$.

Close the loop.



$$\int_{-R}^R \frac{dx}{(1+x^2)^2} + \int_{C_R} \frac{dz}{(1+z^2)^2} = 2\pi i \text{Res}(\cdot, i)$$

Since $\left| \int_{C_R} \frac{dz}{(1+z^2)^2} \right| \leq \frac{1}{(R^2-1)^2} \cdot \pi R \rightarrow 0 \text{ as } R \rightarrow \infty.$

$$|1+z^2| \geq |z|^2 - 1, \quad |z| = R \gg 1$$

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{dz}{(1+z^2)^2} \right| = 0.$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2\pi i \text{Res}(\cdot, i) = \int_{C_\epsilon} \frac{dz}{(z-i)^2(z+i)^2}$$

$$= 2\pi i g'(i), \quad g = \frac{1}{(z+i)^2}, \quad \text{Cauchy deriv formula}$$

$$= (2\pi i) \frac{(-2)}{(2i)^3} \quad g' = \frac{-2}{(z+i)^3}$$

$$= \pi/2.$$

Frequent Intermediate step:

$$\left| \int_{C_R} \frac{P(z)}{Q(z)} dz \right| \leq \left(\frac{\max_{C_R} |P|}{\min_{C_R} |Q|} \right) (\text{Length } C_R).$$

e.g. $C_R = \{ |z| = R \}$

$$\left| \int_{C_R} \frac{1+z}{z^5+2z+1} dz \right| \leq \frac{(1+R)}{R^5-2R-1} \cdot 2\pi R \rightarrow 0 \text{ as } R \rightarrow \infty,$$

Note: $|z^5+2z+1| \geq |z|^5 - |2z+1| \geq |z|^5 - 2|z| - 1.$

$$\Rightarrow \min_{C_R} |z^5+2z+1| \geq R^5 - 2R - 1.$$

reverse triangle ineq (*)
 $|a+b| \geq |a| - |b|$

triangle ineq
 $-|a+b| \geq -|a| - |b|$

(*)
Note: prev reverse triangle
 $|a-b| \geq ||a| - |b||$
 $\geq |a| - |b|$
replace $b \leftrightarrow -b$.