

# Residue Theorem Reloaded

Remark: If  $f(z)$  has a pole of order  $k$  at  $z_0$ , then:

$$\text{Res}(f, z_0) = \frac{1}{(k-1)!} \lim_{z \rightarrow z_0} \left( \frac{d^{k-1}}{dz^{k-1}} \left( (z-z_0)^k f(z) \right) \right).$$

Proof:  $f(z) = a_k (z-z_0)^{-k} + \dots + a_{-1} (z-z_0)^{-1} + a_0 + a_1 (z-z_0) + \dots$

$$\Rightarrow \frac{d^{k-1}}{dz^{k-1}} \left( (z-z_0)^k f \right) = \frac{d^{k-1}}{dz^{k-1}} \left( a_{-k} + \dots + a_{-1} (z-z_0)^{k-1} + a_0 (z-z_0)^k + \dots \right)$$

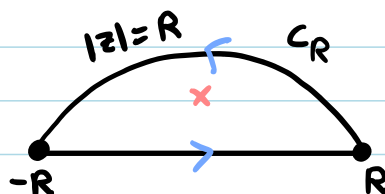
$$= (k-1)! a_{-1} + (k(k-1)\dots 2 a_0)(z-z_0) + \dots$$

$$\rightarrow (k-1)! a_{-1} \quad \text{as } z \rightarrow z_0. \quad \square$$

ex) Calculate  $I = \int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5} dx.$

Consider

$$\int_{C_R} \frac{z e^{i\pi z}}{z^2 + 2z + 5} dz.$$



$\frac{z \sin \pi z}{z^2 + 2z + 5}$   
will not work.

Need:

$$(*)_R \rightarrow 0.$$

Poles at:  $z^2 + 2z + 5 = 0$

$$z = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i. \quad \text{Only } -1 + 2i \text{ inside loop.}$$

$$\int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{x^2 + 2x + 5} dx + \lim_{R \rightarrow \infty} \underbrace{\int_{C_R} \frac{z e^{i\pi z}}{z^2 + 2z + 5} dz}_{(*)_R} = 2\pi i \text{Res}(\cdot, -1 + 2i).$$

Note:  $\lim_{R \rightarrow \infty} (*)_R = 0$ , since:

$$\left| \int_{C_R} \frac{z e^{i\pi z}}{z^2 + 2z + 5} dz \right| \leq \max_{C_R} \left( \frac{|z|}{|z|^2 - 2|z| - 5} \right) \cdot \frac{\pi}{\pi}. \quad \text{This is by Jordan's Lemma}$$

Jordan's Lemma, to be proved below, states

$$\left| \int_{C_R} f(z) e^{i\pi z} dz \right| \leq \max_{C_R} |f|.$$

$$|(*)_R| \leq \frac{R}{R^2 - 2R - 5} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

$$\therefore \int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{x^2 + 2x + 5} dx = 2\pi i \operatorname{Res}(\cdot, -1 + 2i).$$

$$= 2\pi i \frac{P(-1+2i)}{Q'(-1+2i)}, \quad \begin{aligned} P &= z e^{i\pi z} \\ Q &= z^2 + 2z + 5 \\ Q' &= 2z + 2 \end{aligned}$$

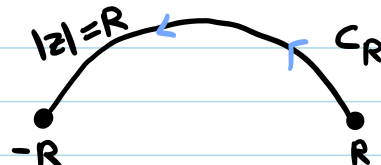
$$= \frac{2\pi i (-1+2i)(e^{i\pi(-1+2i)})}{2(-1+2i) + 2}$$

$$= \frac{\pi}{2} (-1+2i)(-1) e^{-2\pi}.$$

$$\operatorname{Im} \int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{x^2 + 2x + 5} dx = -\pi e^{-2\pi} \Rightarrow I = -\pi e^{-2\pi}.$$

Jordan's Lemma: Let  $C_R$  be the upper semi-circle of radius  $R > 0$ . Let  $\beta > 0$ . Then:

$$\left| \int_{C_R} f(z) e^{i\beta z} dz \right| \leq \frac{\pi}{\beta} \max_{C_R} |f|.$$



Proof: Parametrize  $C_R$  by:  
 $z(t) = R e^{it}, \quad 0 \leq t \leq \pi.$

$$\int_{C_R} f(z) e^{i\beta z} dz = \int_0^\pi f(z(t)) e^{i\beta x(t)} e^{-\beta y(t)} R i e^{it} dt.$$

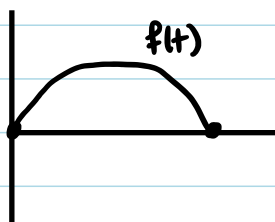
$$\begin{aligned}
 \left| \int_{C_R} f e^{ipz} dz \right| &\leq R \int_0^\pi |f(z(t))| e^{-\beta y(t)} dt \\
 &= R \max_{C_R} |f| \int_0^\pi e^{-\beta R \sin t} dt \quad y(t) = R \sin t \\
 &= R \max_{C_R} |f| \left( \int_0^{\pi/2} e^{-\beta R \sin t} dt + \int_{\pi/2}^\pi e^{-\beta R \sin t} dt \right)
 \end{aligned}$$

Claim:  $\begin{cases} \sin x \geq \frac{2x}{\pi} & \text{on } [0, \pi/2] \\ \sin x \geq \frac{2(\pi-x)}{\pi} & \text{on } [\pi/2, \pi]. \end{cases}$

$$\begin{aligned}
 &\leq R \max_{C_R} |f| \left( \int_0^{\pi/2} e^{-\frac{2\beta R}{\pi} t} dt + \int_{\pi/2}^\pi e^{\beta R (\frac{2}{\pi})(x-\pi)} dt \right) \\
 &= R \max_{C_R} |f| \left( \left[ \frac{(-\pi)}{2\beta R} e^{-\frac{2\beta R}{\pi} t} \right]_0^{\pi/2} + \left[ \frac{1}{\beta R (\frac{2}{\pi})} e^{\beta R (\frac{2}{\pi})(x-\pi)} \right]_{\pi/2}^\pi \right) \\
 &= \max_{C_R} |f| \left( -\frac{\pi}{\beta} e^{-\beta R} + \frac{\pi}{\beta} \right).
 \end{aligned}$$

Proof of claim:

$f(t) = \sin t - \frac{2}{\pi} t$  solves:  $f(0) = 0$   
 $f(\pi/2) = 0$   
 $f''(t) < 0$  over  $[0, \pi/2]$ .



$\Rightarrow f(t) \geq 0$  on  $[0, \pi/2]$ .

Similar argument:  $f(t) = \sin t - \frac{2}{\pi}(\pi - t) \geq 0$  on  $[\pi/2, \pi]$ .

□

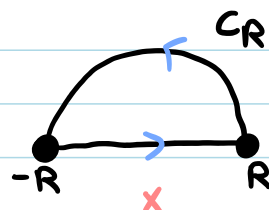
ex) Calculate  $I = \int_{-\infty}^{\infty} \frac{\sin x}{x+i} dx$ . Write  $\sin x = \frac{1}{2i} (e^{ix} - e^{-ix})$ .

$$I = \underbrace{\frac{1}{2i} \int_{-\infty}^{\infty} \frac{e^{ix}}{x+i} dx}_{I_1} - \underbrace{\frac{1}{2i} \int_{-\infty}^{\infty} \frac{e^{-ix}}{x+i} dx}_{I_2}.$$

Calculate  $I_1$ :

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x+i} dx + \lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{iz}}{z+i} dz = 0$$

no poles enclosed



$$\left| \int_{C_R} \frac{e^{iz}}{z+i} dz \right| \leq \frac{\pi}{R-1} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

Jordan Lem

$$|z+i| \geq |z| - |i|$$

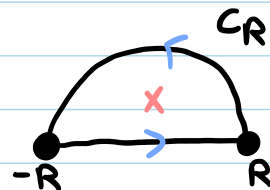
$$\frac{1}{|z+i|} \leq \frac{1}{|z| - 1}$$

$$\Rightarrow I_1 = 0.$$

Calculate  $I_2$ : replace  $x \leftrightarrow -x$  to setup for Jordan's lemma.

$$I_2 = \int_{x=-\infty}^{x=\infty} \frac{e^{-ix}}{x+i} dx = \int_{u=\infty}^{u=-\infty} \frac{e^{iu}}{-u+i} (-du) = \int_{-\infty}^{\infty} \frac{e^{ix}}{i-x} dx.$$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{i-x} dx + \lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{iz}}{i-z} dz = 2\pi i \operatorname{Res}(\cdot, i).$$



$$\left| \int_{C_R} \frac{e^{iz}}{i-z} dz \right| \leq \frac{\pi}{R-1} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

$$\therefore I_2 = 2\pi i \operatorname{Res}\left(\frac{e^{iz}}{i-z}, i\right) = (2\pi i) \frac{e^{i^2}}{-1} = -2\pi e^{-1} i.$$

$$I = (2i)^{-1} I_1 - (2i)^{-1} I_2 \Rightarrow I = \pi/e.$$

Note: To apply Jordan's lemma to  $I = \int_{-\infty}^{\infty} f(x) e^{-i\beta x} dx$ ,  $\beta > 0$ , switch  $x \leftrightarrow -x$ .

$$I = \int_{-\infty}^{\infty} f(-x) e^{i\beta x} dx.$$

ex) Calculate  $I = \int_0^{\infty} \frac{\sin x}{x} dx$ .

Note:  $I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x} dx$  since  $\frac{\sin x}{x}$  is even.

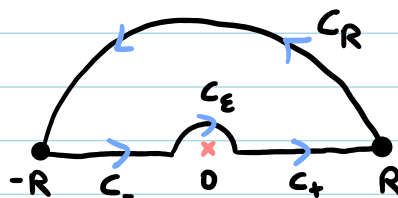
$$I = \frac{1}{2} \operatorname{Im} \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx.$$

$$\int_C \frac{e^{iz}}{z} dz = 0, \text{ where } C = C_R + C_- + C_\epsilon + C_+.$$

①  $\left| \int_{C_R} \frac{e^{iz}}{z} dz \right| \leq \pi R^{-1}$

Jordan  
Lem

$$\rightarrow 0 \text{ as } R \rightarrow \infty.$$



②  $\int_{C_\epsilon} \frac{e^{iz}}{z} dz = \int_{\pi}^0 \frac{e^{iz(t)}}{\epsilon e^{it}} i \epsilon e^{it} dt, \quad z(t) = \epsilon e^{it}.$

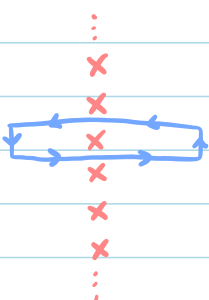
$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{C_\epsilon} \frac{e^{iz}}{z} dz &= \lim_{\epsilon \rightarrow 0} \int_{\pi}^0 i e^{i\epsilon e^{it}} dt \\ &= i \int_{\pi}^0 dt = -i\pi. \end{aligned}$$

$$\int_{C_R + C_- + C_\epsilon + C_+} = 0 \Rightarrow \int_{-\infty}^0 \frac{e^{ix}}{x} dx - i\pi + \int_0^{\infty} \frac{e^{ix}}{x} dx = 0.$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = i\pi.$$

$$\Rightarrow \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

ex) Calculate  $I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$ ,  $0 < a < 1$ .

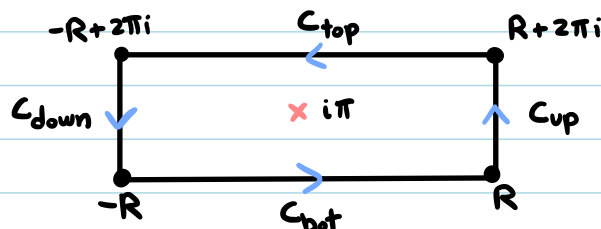


$z = i\pi + i(2\pi k)$   
 $e^z = -1$

Consider  $\int_C \frac{e^{az}}{1+e^z} dz$ , where  $C = C_{\text{down}} + C_{\text{bot}} + C_{\text{up}} + C_{\text{top}}$ .

$\int_C \frac{e^{az}}{1+e^z} dz = 2\pi i \operatorname{Res}(\cdot, i\pi)$

Since poles @  $z = i\pi + 2\pi i k$ .



$= 2\pi i \frac{P(i\pi)}{Q'(i\pi)} \quad \begin{matrix} P = e^{az} \\ Q = 1 + e^z \end{matrix}$

$= -2\pi i e^{ia\pi}$

$\left| \int_{C_{\text{up}}} \frac{e^{az}}{1+e^z} dz \right| \leq \max_{C_{\text{up}}} \left| \frac{e^{az}}{1+e^z} \right| \text{Length}(C_{\text{up}})$

$\leq \frac{e^{aR}}{e^R - 1} (2\pi) \rightarrow 0 \text{ as } R \rightarrow \infty, \text{ since } 0 < a < 1.$

$\left| \int_{C_{\text{down}}} \frac{e^{az}}{1+e^z} dz \right| \leq \frac{e^{-aR}}{1 - e^{-R}} (2\pi) \rightarrow 0 \text{ as } R \rightarrow \infty.$

$\int_{C_{\text{top}}} \frac{e^{az}}{1+e^z} dz = \int_R^{-R} \frac{e^{ax + 2\pi ia}}{1 + e^{x + 2\pi i}} dx = -e^{2\pi ia} \int_{-R}^R \frac{e^{ax}}{1 + e^x} dx.$

$\int_C = -2\pi i e^{ia\pi} \Rightarrow I - e^{2\pi ia} I = -2\pi i e^{ia\pi}$

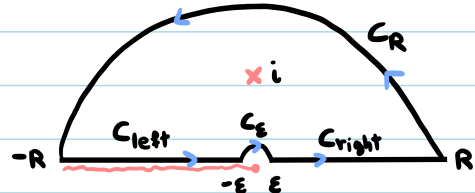
$\Rightarrow I = \frac{-2\pi i}{e^{-i\pi a} - e^{i\pi a}}$

$\Rightarrow I = \frac{\pi}{\sin \pi a}$

ex) Calculate  $\int_0^{\infty} \frac{\ln x}{x^2+1} dx$ .

Consider  $\int_C \frac{\text{Log } z}{z^2+1} dz$ , where  $C = C_{\text{left}} + C_{\epsilon} + C_{\text{right}} + C_R$ .

$$\begin{aligned} \int_C \frac{\text{Log } z}{z^2+1} dz &= 2\pi i \text{Res}(\cdot, i) \\ &= 2\pi i \frac{\text{Log } i}{2i} \\ &= \pi (\ln 1 + i\pi/2) \\ &= i\pi^2/2. \end{aligned}$$



$$\left| \int_{C_R} \frac{\text{Log } z}{z^2+1} dz \right| \leq \frac{\ln R + \pi}{R^2-1} \cdot \pi R \rightarrow 0 \text{ as } R \rightarrow \infty.$$

$$\left| \int_{C_{\epsilon}} \frac{\text{Log } z}{z^2+1} dz \right| \leq \frac{|\ln \epsilon| + \pi}{1-\epsilon^2} \cdot \pi \epsilon \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

$$\begin{aligned} \int_{C_{\text{left}}} \frac{\text{Log } z}{z^2+1} dz &= \int_R^{\epsilon} \frac{\text{Log}(te^{i\pi})}{1+t^2} e^{i\pi} dt \\ &\xrightarrow[\substack{\epsilon \rightarrow 0 \\ R \rightarrow \infty}]{} \int_0^{\infty} \frac{(\ln t + i\pi)}{1+t^2} dt \end{aligned}$$

$$\begin{aligned} z(t) &= e^{i\pi} t, \quad dz = e^{i\pi} dt \\ \text{Log } z(t) &= \text{Log}(te^{i\pi}). \end{aligned}$$

$$\therefore \int_0^{\infty} \frac{\ln x}{1+x^2} dx + \int_0^{\infty} \frac{i\pi}{1+x^2} dx + \int_0^{\infty} \frac{\ln x}{1+x^2} dx = i\frac{\pi^2}{2}.$$

Real part of above equation:  $\int_0^{\infty} \frac{\ln x}{1+x^2} dx = 0.$

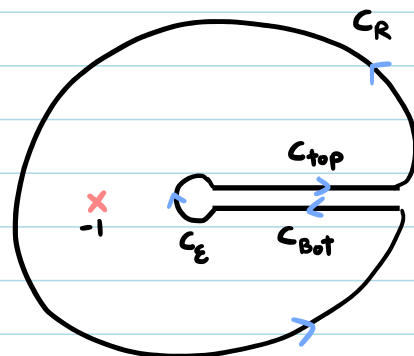
Note:  $|\text{Log } z| \leq |\ln |z|| + \pi$  since:  $|\text{Log } z| = |\ln |z| + i\varphi|$ ,  
 $z = |z|e^{i\varphi}$ ,  $-\pi < \varphi \leq \pi$ .

ex) Compute  $\int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx$  where  $0 < \alpha < 1$ .

Consider  $\int_C \frac{z^{\alpha-1}}{1+z} dz$ , with  $C = C_{\text{Bot}} + C_{\epsilon} + C_{\text{top}} + C_R$ ,

and choice of branch

$$z^{\alpha-1} = |z|^{\alpha-1} e^{i(\alpha-1)\varphi}, \quad z = |z| e^{i\varphi} \\ 0 \leq \varphi < 2\pi.$$



Note: not principal branch.

Branch cut at  $\varphi = 0$ : positive real axis.

$$\int_C \frac{z^{\alpha-1}}{1+z} dz = 2\pi i \operatorname{Res}(\cdot, -1) = 2\pi i e^{i(\alpha-1)\pi}.$$

$$\left| \int_{C_R} \frac{z^{\alpha-1}}{1+z} dz \right| \leq \frac{R^{\alpha-1}}{R-1} \pi R \rightarrow 0 \text{ as } R \rightarrow \infty \quad (0 < \alpha < 1)$$

$$\left| \int_{C_{\epsilon}} \frac{z^{\alpha-1}}{1+z} dz \right| \leq \frac{\epsilon^{\alpha-1}}{1-\epsilon} \pi \epsilon \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

$$\int_{C_{\text{top}}} \frac{z^{\alpha-1}}{1+z} dz = \int_{\epsilon}^R \frac{x^{\alpha-1}}{1+x} dx \rightarrow \int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx.$$

$$\int_{C_{\text{Bot}}} \frac{z^{\alpha-1}}{1+z} dz = \int_R^{\epsilon} \frac{t^{\alpha-1}}{1+t} e^{i2\pi(\alpha-1)} dt \rightarrow -e^{i2\pi(\alpha-1)} \int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx.$$

$$\begin{array}{c} \text{---} \epsilon \quad \leftarrow \quad \leftarrow \quad \leftarrow \quad R \text{---} \\ z(t) = e^{2\pi i} t \end{array}$$

$$\therefore (1 - e^{i2\pi(\alpha-1)}) \int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx = 2\pi i e^{i(\alpha-1)\pi}.$$

Mult by  $e^{-i\alpha\pi}$

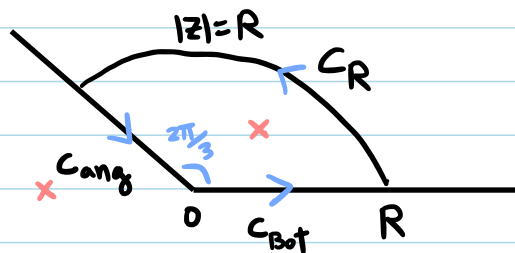
$$\therefore \int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx = \frac{-2\pi i}{e^{-i\alpha\pi} - e^{i\alpha\pi}} = \frac{\pi}{\sin(\alpha\pi)}.$$

ex) Compute  $\int_0^{\infty} \frac{dx}{x^3+1}$ .

Consider  $\int_C \frac{dz}{z^3+1}$ , where  $C = C_{\text{Bot}} + C_{\text{ang}} + C_R$ .

$$\int_C \frac{dz}{z^3+1} = 2\pi i \operatorname{Res}(\cdot, e^{i\pi/3})$$

$$= \frac{2\pi i}{3} e^{-i2\pi/3}.$$



$$\left| \int_{C_R} \frac{dz}{z^3+1} \right| \leq \frac{1}{R^3-1} \cdot \pi R \rightarrow 0 \text{ as } R \rightarrow \infty.$$

$$z^3 = -1$$

$$z = e^{i\pi/3}, -1, e^{i5\pi/3}$$

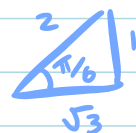
On  $-C_{\text{ang}}$ ,  $z = t e^{2\pi i/3}$ ,  $0 \leq t \leq R$ .

$$\int_{-C_{\text{ang}}} \frac{dz}{z^3+1} = \int_0^R \frac{e^{2\pi i/3}}{t^3+1} dt \xrightarrow{R \rightarrow \infty} e^{2\pi i/3} \int_0^{\infty} \frac{dx}{x^3+1}.$$

$$\therefore -e^{2\pi i/3} \int_0^{\infty} \frac{dx}{x^3+1} + \int_0^{\infty} \frac{dx}{x^3+1} = \frac{2\pi i}{3} e^{-i2\pi/3}.$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{x^3+1} = \frac{2\pi i}{3} \frac{e^{-i2\pi/3}}{1 - e^{i2\pi/3}} \quad \text{multiply by } \frac{e^{-i\pi/3}}{e^{-i\pi/3}}$$

$$= \frac{2\pi i (-1)}{3 (e^{-i\pi/3} - e^{i\pi/3})} = \frac{\pi}{3 \sin(\pi/3)} = \frac{2\pi}{3\sqrt{3}}.$$



Note: Textbook uses notation: p.v.  $\int_{-\infty}^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$ .

This is helpful when  $\int_{-\infty}^{\infty} f(x) dx \neq \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx$ , e.g.

$$\lim_{R \rightarrow \infty} \int_{-R}^R x dx \neq \lim_{R \rightarrow \infty} \int_{-R}^0 x dx + \lim_{R \rightarrow \infty} \int_0^R x dx.$$