

Argument Principle

Def: A function f is meromorphic in $D \subseteq \mathbb{C}$ if at each $z_0 \in D$, $\exists r > 0$ st. on $0 < |z - z_0| < r$ then

$$f(z) = \sum_{k=-m}^{\infty} a_k (z - z_0)^k \quad \text{for some } m \in \mathbb{Z}_{\geq 0}.$$

Meromorphic = analytic except with some poles.

ex) $f(z) = z + z^{-2}$ is meromorphic

ex) $e^{1/z}$ is not meromorphic

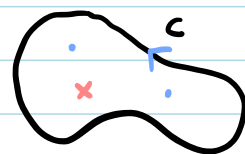
ex) P/Q , with P, Q polynomials, is meromorphic.

Thm: (Argument Principle)

Let f be meromorphic on and inside a simple closed curve C . Suppose f has no zeros or poles on C itself.

$$\int_C \frac{f'(z)}{f(z)} dz = 2\pi i \left(\# \text{zeros in } C - \# \text{poles in } C \right).$$

ccw →



Note: multiplicities included, e.g. 1 zero of order 3 counts as 3.

Proof: Let $\{p_i\}$ be the zeros and poles in C .

Let C_i be a small circle around p_i .

$$\text{Green's Thm} \Rightarrow \int_C \frac{f'(z)}{f(z)} dz = \sum_i \int_{C_i} \frac{f'(z)}{f(z)} dz.$$

Let $p_i = z_0$ be a zero of order m . Near z_0 :

$$f(z) = a_m (z - z_0)^m + a_{m+1} (z - z_0)^{m+1} + a_{m+2} (z - z_0)^{m+2} + \dots$$

$$f'(z) = m a_m (z - z_0)^{m-1} + (m+1) a_{m+1} (z - z_0)^m + \dots$$

$$\frac{f'}{f} = \frac{m a_m (z - z_0)^{m-1} \left(1 + \frac{m+1}{m} \frac{a_{m+1}}{a_m} (z - z_0) + \dots \right)}{a_m (z - z_0)^m \left(1 + \frac{a_{m+1}}{a_m} (z - z_0) + \dots \right)}$$

$$= \frac{m}{(z - z_0)} \left(1 + b_1 (z - z_0) + b_2 (z - z_0)^2 + \dots \right) \left(1 + c_1 (z - z_0) + c_2 (z - z_0)^2 + \dots \right)^{-1}$$

$$\frac{f'}{f} = \frac{m}{z-z_0} + \sum_{n=0}^{\infty} d_n (z-z_0)^n.$$

$$\Rightarrow \int_{C_i} \frac{f'}{f} dz = (2\pi i)m, \quad \text{since} \quad \int_C (z-z_0)^k dz = \begin{cases} 2\pi i & k=-1 \\ 0 & \text{else} \end{cases}$$

Similarly: if z_0 is a pole of order m ,

$$f(z) = a_{-m}(z-z_0)^{-m} + a_{-m+1}(z-z_0)^{-m+1} + \dots$$

$$\frac{f'}{f} = \frac{-m}{z-z_0} + \sum_{n=0}^{\infty} d_n (z-z_0)^n.$$

$$\int_{C_i} \frac{f'}{f} dz = -(2\pi i)m.$$

$$\therefore \int_C \frac{f'(z)}{f(z)} dz = \sum_{\text{zeros}} (2\pi i)m_j - \sum_{\text{poles}} (2\pi i)m_j. \quad \square$$

Thm: (Rouché's Theorem)

Let C be a simple closed curve.

Suppose f and g are analytic on and inside C .

Suppose $|g(z)| < |f(z)|$ on C .

Then: f and $f+g$ have the same number of zeros inside C .
(counted with multiplicity)

Proof: Let $h(z) = \frac{f(z)+g(z)}{f(z)}$. Note f cannot be 0 on C : $|g(z)| < 0$ impossible

$$\begin{aligned} \text{Arg princ} \Rightarrow \int_C \frac{h'}{h} dz &= 2\pi i \left(\begin{array}{c} \# \text{ zeros} \\ \text{of } h \end{array} - \begin{array}{c} \# \text{ poles} \\ \text{of } h \end{array} \right) \\ &= 2\pi i \left(\begin{array}{c} \# \text{ zeros} \\ \text{of } f+g \end{array} - \begin{array}{c} \# \text{ zeros} \\ \text{of } f \end{array} \right). \end{aligned}$$

Claim: $\int_C \frac{h'}{h} dz = 0$, which finishes the proof.

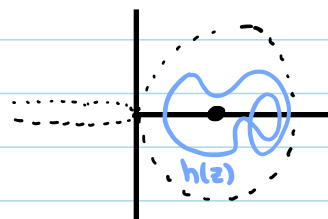
Proof of claim: $|h(z)-1| = \frac{f+g}{f} - 1 = \left| \frac{g}{f} \right| < 1$ on C .

$\Rightarrow |h(z)-1| < 1$ inside C by max princ.

$\therefore \text{Log } h(z)$ is analytic on and inside C .

$$\therefore \int_C \frac{h'}{h} dz = \underset{\text{FTC}}{\text{Log } h(z_f)} - \underset{z_f = z_i \text{ on loop } C}{\text{Log } h(z_i)} = 0.$$

□



ex) Prove $z^5 + z + 1$ has 5 zeros in $|z| < 2$.

$$\text{Let } f(z) = z^5 \\ g(z) = z + 1.$$

$$\text{On } |z|=2, |g(z)| \leq |z| + 1 \leq 2 + 1 \leq 2^5 = |z|^5 = |f(z)|.$$

$\therefore z^5 + z + 1$ and z^5 have same # zeros in $|z| < 2$.

Note: z^5 has a zero of order 5 at $z_0 = 0$.

$\therefore z^5 + z + 1$ has "5 zeros" (counted with multiplicity) in $|z| < 2$.

ex) Find the number of zeros of $z^4 + 4z + 1$ in $|z| < 1$.

$$\text{On } |z|=1, \text{ we have: } |z^4 + 1| \leq |z|^4 + 1 = 2, \\ |4z| = 4.$$

$\Rightarrow z^4 + 1 + 4z$ and $4z$ have the same # zeros in $|z| < 1$.

$\Rightarrow z^4 + 1 + 4z$ has 1 zero in $|z| < 1$.

Thm: (Argument Principle v2)

Let f be meromorphic on and inside a simple closed curve C . Suppose f has no zeros or poles on C itself.

$$\begin{aligned} \int_C \frac{f'(z)}{f(z)} dz &= i \left(\Delta_C \arg(f) \right) \\ &:= i \left(\text{total change in argument of } f(z) \text{ along } C \right) \\ &= 2\pi i \left(\# \text{ zeros} - \# \text{ poles} \right) \end{aligned}$$

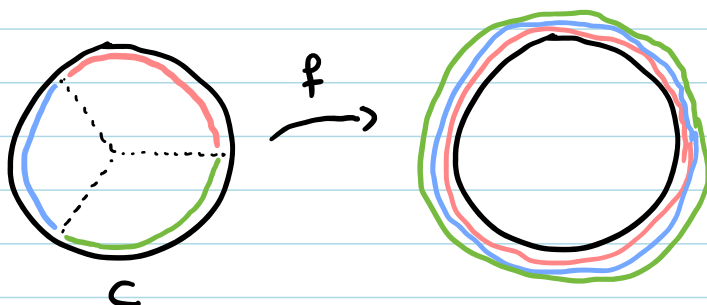


$$\Rightarrow \Delta_C \arg(f) = 2\pi (\# \text{ zeros} - \# \text{ poles}).$$

ex) $f(z) = z^3$.

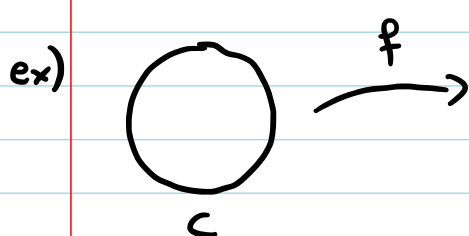
$$C = \{ |z| = 1 \}$$

zeros in $C = 3$.

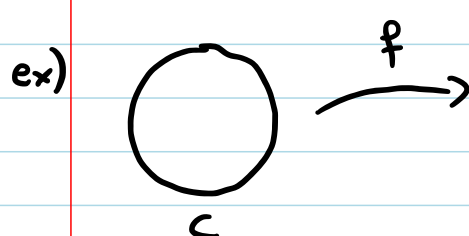


$f(C)$ goes around the circle 3 times

$$\begin{aligned} \Delta_C \arg(f) &= 3 \cdot 2\pi \\ &= 6\pi. \end{aligned}$$



$$\Delta_C \arg(f) = 2 \cdot 2\pi$$



$$\Delta_C \arg(f) = 0.$$

Proof of Argument Principle v2:

Write $\frac{f'(z)}{f(z)} = \frac{d}{dz} \text{Log } f(z) = \frac{d}{dz} \left(\ln |f(z)| + i \text{Arg}(f(z)) \right)$

Write $C = C_1 + \dots + C_N$ where $\arg f$ does not change by more than 2π along C_k .

Choose branch of $\log$ s.t. $z \mapsto \text{Log}^{(k)} f(z)$ is analytic along C_k .

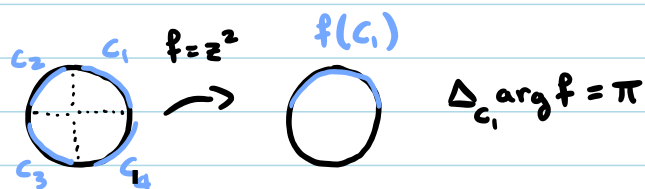
$$\int_C \frac{f'}{f} dz = \sum_k \int_{C_k} \frac{d}{dz} \text{Log}^{(k)} f(z) dz$$

$$= \sum_k \underbrace{\ln |f(z_{k+1})| - \ln |f(z_k)|}_{\text{cancel as telescoping sum}} + i \left(\text{Arg}^{(k)} f(z_{k+1}) - \text{Arg}^{(k)} f(z_k) \right)$$

$$= i \sum_k \Delta_{C_k} \arg f$$

$$= i \Delta_C \arg f.$$

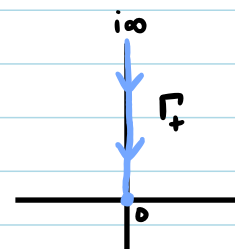
□



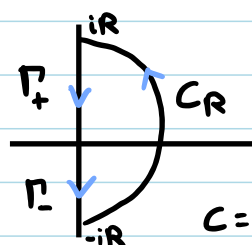
$$C = C_1 + C_2 + C_3 + C_4, \quad \Delta_C \arg f = \pi + \pi + \pi + \pi = 2 \cdot 2\pi.$$

Thm: (Nyquist) Let $P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0$, with $a_i \in \mathbb{R}$ and $P(0) > 0$.

$$\left(\begin{array}{l} \# \text{ zeros of } P \\ \text{with } \text{Re } z > 0 \end{array} \right) = \frac{1}{2\pi} \left(n\pi + 2 \Delta_{\Gamma_+} \arg(P) \right).$$



Proof: Consider the semi-circle loop:



$$2\pi \left(\begin{array}{l} \# \text{ zeros} \\ \text{in } C \end{array} \right) = \Delta_{C_R} \arg P + \Delta_{\Gamma_+} \arg P + \Delta_{\Gamma_-} \arg P. \quad (*)$$

$$C = C_R + \Gamma_+ + \Gamma_-.$$

Note: $\Delta_{\Gamma_+} \arg P = \Delta_{\Gamma_-} \arg P$, since:

Same proof as
arg princ

(1) $\Delta_C \arg f = \operatorname{Im} \int_C \frac{f'}{f} dz$ for any curve C (not nec loop)
with no zeros of f on C .



$$(2) \Delta_{\Gamma_+} \arg P = \operatorname{Im} \int_R^0 \frac{P'(it)}{P(it)} i dt = -\operatorname{Im} \int_R^0 \frac{P'(it)}{P(it)} i dt$$

$$= -\operatorname{Im} \int_R^0 \frac{P'(-it)(-i) dt}{P(-it)}$$



$$(3) \Delta_{\Gamma_-} \arg P = \operatorname{Im} \int_0^R \frac{P'(-it)}{P(-it)} (-i) dt \quad \checkmark \quad z(t) = -it.$$

Take limit of (*):

$$2\pi \left(\begin{array}{c} \# \text{ zeros with} \\ \operatorname{Re} > 0 \end{array} \right) = \lim_{R \rightarrow \infty} \Delta_{C_R} \arg P + 2 \Delta_{\Gamma_+} \arg P.$$

Remains to prove:

$$\lim_{R \rightarrow \infty} \Delta_{C_R} \arg P = n\pi.$$

$$\Delta_{C_R} \arg P = \operatorname{Im} \int_{C_R} \frac{P'(z)}{P(z)} dz, \quad \text{by (1)}$$

$$= \operatorname{Im} \int_{C_R} \frac{n z^{n-1} + (n-1)a_{n-1} z^{n-2} + \dots + a_1}{z^n + a_{n-1} z^{n-1} + \dots + a_0} dz$$

$$= \operatorname{Im} \int_{-\pi/2}^{\pi/2} \frac{n R^{n-1} (e^{i(n-1)t} + R^{-1}(\dots))}{R^n (e^{int} + R^{-1}(\dots))} R i e^{it} dt$$

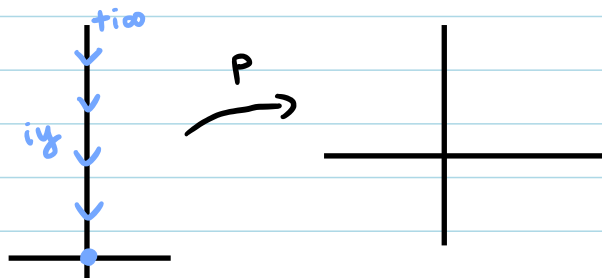
$$= \operatorname{Im} i \int_{-\pi/2}^{\pi/2} n \left(\frac{1 + R^{-1}(\dots)}{1 + R^{-1}(\dots)} \right) dt$$

$$\rightarrow n\pi \quad \text{as } R \rightarrow \infty.$$

□

ex) Prove there are no zeros of $z^3 + 2z^2 + z + 1 = 0$ with $\operatorname{Re} z > 0$.

$$\# \text{ such zeros} = (2\pi)^{-1} (3\pi + 2 \Delta_{\Gamma} \arg P).$$

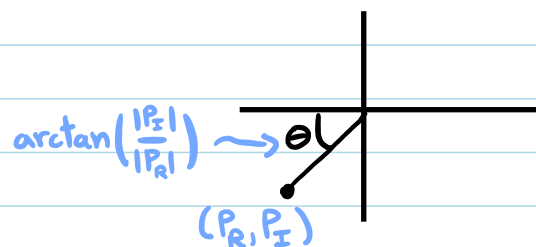


$$P(iy) = (iy)^3 + 2(iy)^2 + iy + 1 \Rightarrow P(iy) = P_R(y) + i P_I(y) \\ = -2y^2 + 1 + i(-y^3 + y)$$

$$P_R = -2y^2 + 1, \quad y > 0. \\ P_I = -y^3 + y$$

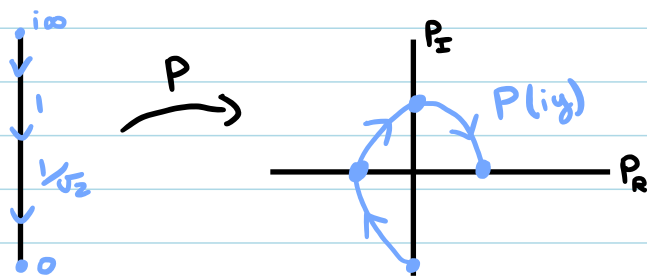
$$\textcircled{1} \arctan \left(\frac{|P_I|}{|P_R|} \right) \xrightarrow{\rightarrow \infty} \frac{\pi}{2} \text{ as } y \rightarrow \infty, \quad \begin{matrix} P_R < 0 \\ P_I < 0 \end{matrix} \text{ as } y \rightarrow \infty$$

$$\textcircled{2} \text{ Set } P_I = 0. \\ -y^3 + y = 0 \Rightarrow y = 0, 1.$$



$$\textcircled{3} \text{ Set } P_R = 0. \\ -2y^2 + 1 = 0 \Rightarrow y = 1/\sqrt{2}.$$

Sketch using $\textcircled{1}\textcircled{2}\textcircled{3}$



$$\therefore \Delta_{\Gamma} \arg P = -\frac{3\pi}{2}. \text{ Minus because CW.}$$

$$\therefore \# \text{ zeros} = \frac{1}{2\pi} \left(3\pi - 2 \cdot \frac{3\pi}{2} \right) = 0.$$

ex) Find the number of zeros of $p(z) = z^3 - 2z^2 + 4$ with $\operatorname{Re} z > 0$.

$$\# \text{ such zeros} = \frac{1}{2\pi} (3\pi + 2\Delta_{\Gamma^+} \arg P)$$

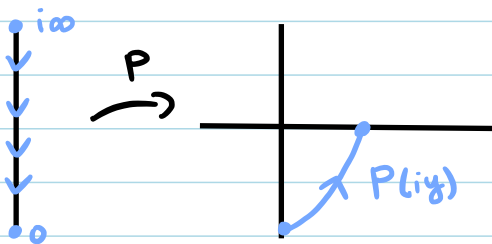
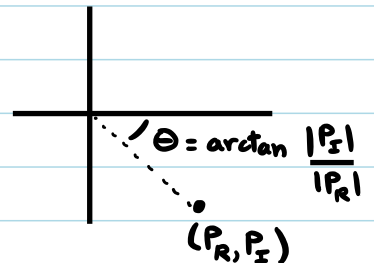
$$P(iy) = (iy)^3 - 2(iy)^2 + 4 \\ = P_R + i P_I$$

$$P_R = 2y^2 + 4, \quad y > 0. \\ P_I = -y^3$$

① $P_R > 0$ as $y \rightarrow \infty$, $\arctan \frac{|P_I|}{|P_R|} \rightarrow \frac{\pi}{2}$ as $y \rightarrow \infty$
 $P_I < 0$

② $P_I = 0$ at $y = 0$

③ $P_R = 0$ nowhere if $y > 0$.



$$\Delta_{\Gamma^+} \arg P = \pi/2.$$

$$\therefore \# \text{ zeros} = \frac{1}{2\pi} (3\pi + 2 \cdot \frac{\pi}{2}) = 2.$$

How this theory appears in engineering:

- ① Understanding a circuit leads to an ODE for current or voltage.

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y = f(t).$$

- ② Taking the Laplace transform creates an analytic function:

$$Y(z) = \int_0^{\infty} y(t) e^{-zt} dt = \mathcal{L}(y) \text{ defined on } \operatorname{Re} z > 0.$$

Theory of analytic functions applies to Laplace transform.

$$\textcircled{3} \mathcal{L}(y^{(n)}) = z^n Y(z) - z^{n-1}y(0) - z^{n-2}y'(0) - \dots - y^{(n-1)}(0).$$

can be proved by integration by parts

$$\mathcal{L}(y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y)$$

$$= (z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0)Y(z) + \left(\begin{array}{l} \text{polynomial deg } n-1 \text{ involving} \\ \text{initial conditions } y(0), y'(0), \dots \text{ etc} \end{array} \right).$$

Take Laplace transform of eqn ①

$$\mathcal{L}(f) = \underbrace{(z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0)}_{P(z)} Y(z) + \left(\begin{array}{l} \text{polynomial involving} \\ \text{initial cond} \end{array} \right)$$

$$\therefore Y(z) = \frac{1}{P(z)} F(z).$$

- ④ Invert Laplace transform to get $y(t)$:

$$y(t) = \sum_k \operatorname{Res} (Y(z) e^{zt}, p_k)$$

$\nwarrow$ poles of $Y(z)e^{zt}$

- ⑤ Suppose $F(z)$ has no poles, so that poles of $Y(z)$ come from zeros of $P(z)$.

If there is $z_0 \in \mathbb{C}$ with $P(z_0) = 0$, $\operatorname{Re} z_0 > 0$, the system is unstable:
e.g. if z_0 is a simple zero:

$$y(t) = \operatorname{Res}\left(\frac{F}{p}, z_0\right) e^{z_0 t} + \dots$$

" $(\operatorname{Re} z_0)t$ $i \operatorname{Im} z_0 t$
 $\underbrace{e}_{\text{blow-up as } t \rightarrow \infty}$ $\underbrace{e}_{\text{oscillation}}$

⑥ Nyquist Theorem on # zeros with $\operatorname{Re} z > 0$ is used to detect if the system can have exponentially growing response.