

Fourier Transform

Let $f: \mathbb{R} \rightarrow \mathbb{C}$ be s.t. $\int_{\mathbb{R}} |f(x)| dx < \infty$.

Define: $\hat{f}: \mathbb{R} \rightarrow \mathbb{C}$

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \quad \text{Fourier Transform}$$

Intuition: $f(x)$ function of time

$\hat{f}(k)$ function of frequency

ex) $f(x) = e^{-x^2/2\sigma}$, Gaussian wavepacket of width $\sigma > 0$.

$$\begin{aligned} 2\pi \hat{f}(k) &= \int_{-\infty}^{\infty} e^{-x^2/2\sigma} e^{-ikx} dx \\ &= \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x+i\sigma^2 k)^2} e^{-\sigma^2 k^2/2} dx. \end{aligned}$$

Note: $\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x+i\sigma^2 k)^2} dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2} dx$, since:

$$\begin{aligned} \frac{d}{da} \int_{-\infty}^{\infty} e^{-b(x+a)^2} dx &= \int_{-\infty}^{\infty} 2(-b(x+a)) e^{-b(x+a)^2} dx \\ &= \int_{-\infty}^{\infty} \frac{d}{dx} e^{-b(x+a)^2} dx = \left[e^{-b(x+a)^2} \right]_{-\infty}^{\infty} = 0. \quad b > 0. \end{aligned}$$

$$\therefore \int_{-\infty}^{\infty} e^{-b(x+a)^2} dx = \int_{-\infty}^{\infty} e^{-bx^2} dx \quad \text{for } a \in \mathbb{R}, b > 0.$$

$$\Rightarrow \hat{f}(k) = \frac{1}{2\pi} e^{-\sigma^2 k^2/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2} dx$$

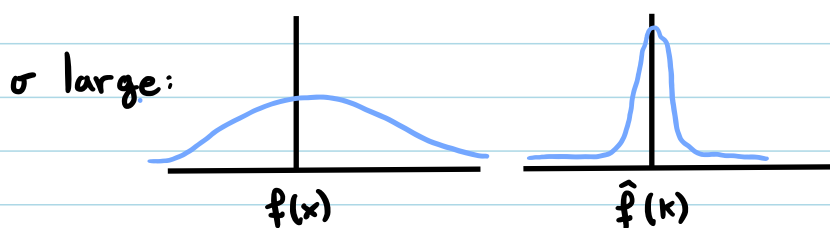
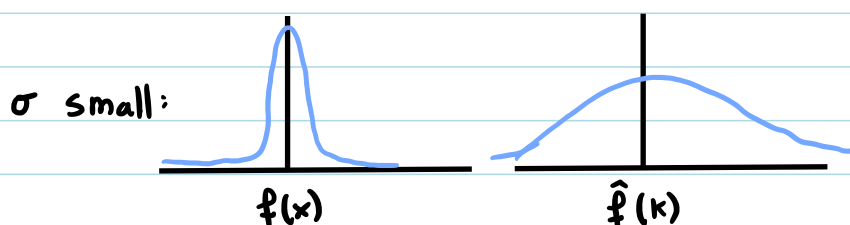
Note: $\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2} dx = \sqrt{2\pi} \sigma$, since this follows from:

Gaussian Integral: Let $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$. Then:

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}, \quad \text{where } \sqrt{a} \text{ denotes the principal branch.}$$

Using this, (*) becomes:

$$\Rightarrow \begin{cases} f(x) = e^{-x^2/2\sigma^2} \\ \hat{f}(k) = (2\pi)^{-1/2} \sigma e^{-\sigma^2 k^2/2} \end{cases}$$

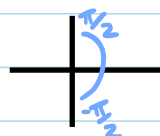


"uncertainty principle"
can't simultaneously
localize a signal both
in space and frequency

Calculation of Gaussian integral

$$\begin{aligned} \left(\int_{-\infty}^{\infty} e^{-ax^2} dx \right)^2 &= \int_{-\infty}^{\infty} e^{-ax^2} dx \int_{-\infty}^{\infty} e^{-ay^2} dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-a(x^2+y^2)} dx dy \\ &= \int_0^{2\pi} \int_0^{\infty} e^{-ar^2} r dr d\theta \\ &= 2\pi \left| -\frac{e^{-ar^2}}{2a} \right|_0^{\infty} = \frac{\pi}{a} (1 - \lim_{R \rightarrow \infty} e^{-aR^2}) = \pi/a. \end{aligned}$$

$$|e^{-aR^2}| = e^{-\operatorname{Re}(a)R^2} \rightarrow 0 \quad \text{since } \operatorname{Re}(a) > 0.$$

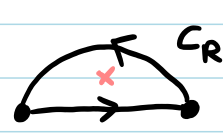
Note: $\operatorname{Re} \int_{-\infty}^{\infty} e^{-ax^2} dx > 0$ since $\operatorname{Re}(a) > 0$. 

$$\Rightarrow \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \quad \text{using principal branch, since } \operatorname{Re} \sqrt{z} \geq 0 \text{ for princ branch.}$$

ex) Find Fourier transform of $f(x) = \frac{1}{1+x^2}$.

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{1+x^2} dx.$$

If $k > 0$: change variables $x \leftrightarrow -x$. $\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^2} dx.$

Close the loop: 

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^2} dx + \lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{ikz}}{1+z^2} dz = 2\pi i \operatorname{Res}(\cdot, i) = \pi e^{-k}.$$

$$\left| \int_{C_R} \frac{e^{ikz}}{1+z^2} dz \right| \stackrel{\text{Jordan's Lemma}}{\leq} \frac{1}{R^2-1} \cdot \pi \rightarrow 0 \text{ as } R \rightarrow \infty.$$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^2} dx = \pi e^{-k} \text{ when } k > 0.$$

$$\text{Next: if } k < 0. \text{ Then } \int_{x=-\infty}^{x=\infty} \frac{e^{-ikx}}{1+x^2} dx = \int_{-\infty}^{\infty} \frac{e^{i|k|x}}{1+x^2} dx = \pi e^{-|k|}.$$

$$\text{Next: if } k=0, \text{ then: } \hat{f}(0) = (2\pi)^{-1} \int_{-\infty}^{\infty} (1+x^2)^{-1} dx = 1/2.$$

$$\Rightarrow \begin{cases} f(x) = (1+x^2)^{-1} \\ \hat{f}(k) = \frac{e^{-|k|}}{2} \end{cases}$$

Def: Inverse Fourier transform of $f: \mathbb{R} \rightarrow \mathbb{C}$ is $\check{f}: \mathbb{R} \rightarrow \mathbb{C}$ given by:

$$\check{f}(x) = \int_{-\infty}^{\infty} f(k) e^{ikx} dk$$

Thm: If f is continuous at x , then $\hat{\check{f}}(x) = f(x)$.
will not prove

Fourier Transform of derivative: for smooth functions with $f(x) \rightarrow 0$ as $x \rightarrow \pm\infty$.

$$\widehat{(f^{(n)})}(k) = (ik)^n \hat{f}(k).$$

Proof:

$$\begin{aligned} (2\pi) \hat{f}'(k) &= \int_{-\infty}^{\infty} f'(x) e^{-ikx} dx \\ &= \int_{-\infty}^{\infty} (ik) f(x) e^{-ikx} dx + \int_{-\infty}^{\infty} f(x) e^{-ikx} \quad \begin{matrix} f \rightarrow 0 \\ \text{as } x \rightarrow \pm\infty \end{matrix} \\ &= ik \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \quad \text{Higher derivatives similar} \end{aligned}$$

Product Formula:

$$\widehat{(f * g)} = (2\pi) \hat{f} \hat{g}, \quad \text{where } (f * g)(x) = \int_{-\infty}^{\infty} f(x-u) g(u) du.$$

Proof:

$$\begin{aligned} (2\pi) \widehat{(f * g)}(k) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-u) g(u) e^{-ikx} dx du, \quad \begin{matrix} x = u+s & u \text{ fixed} \\ dx = ds \end{matrix} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s) g(u) e^{-ik(u+s)} ds du \\ &= \int_{-\infty}^{\infty} f(s) e^{-iks} ds \int_{-\infty}^{\infty} g(u) e^{-iku} du = (2\pi)^2 \hat{f}(k) \hat{g}(k). \end{aligned}$$

ex) Solve $u'' - \omega^2 u = -f(x)$, for $\omega > 0$, $f(x)$ given.
 $u(\pm\infty) = 0$

$$\begin{aligned} \hat{u}'' - \omega^2 \hat{u} &= -\hat{f} \\ (ik)^2 \hat{u} - \omega^2 \hat{u} &= -\hat{f} \end{aligned} \Rightarrow \hat{u}(k) = \frac{\hat{f}(k)}{k^2 + \omega^2}$$

$$\Rightarrow u(x) = \left(\hat{f} \cdot \frac{1}{\omega^2(1 + \frac{k^2}{\omega^2})} \right)^\vee$$

$$= \frac{1}{2\pi\omega^2} f * \left(\frac{1}{1 + \frac{k^2}{\omega^2}} \right)^\vee$$

Compute $\left(\frac{1}{1 + \frac{k^2}{\omega^2}} \right)^\vee$.

Note: $\frac{1}{1+x^2} = \frac{e^{-|x|}}{2} \Rightarrow (2\pi)^{-1} \int_{-\infty}^{\infty} \frac{e^{-iau}}{1+u^2} du = \frac{e^{-|a|}}{2}$

$$\Rightarrow (2\pi)^{-1} \int_{-\infty}^{\infty} \frac{e^{ia\omega u}}{1+u^2} du = \frac{e^{-|a|\omega}}{2}, \quad u = \omega^{-1}k$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{iak}}{1+(\omega^{-1}k)^2} \omega^{-1} dk = e^{-|a|\omega} \pi$$

$$\Rightarrow \frac{1}{1+(\omega^{-1}k)^2}^\vee = \omega \pi e^{-\omega|x|}$$

$$\therefore u(x) = \frac{1}{2\omega} \int_{-\infty}^{\infty} f(x-u) e^{-\omega|u|} du.$$

ex) Solve $\partial_t u = \partial_x \partial_x u$, where $f(x) \rightarrow 0$ as $x \rightarrow \pm \infty$.
 $u(x, 0) = f(x)$

Take Fourier transform in x -variable.

$$\hat{u}(k, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x, t) e^{-ikx} dx.$$

$$\partial_t \hat{u} = (ik)^2 \hat{u} \Rightarrow \partial_t \hat{u} = -k^2 \hat{u} \Rightarrow \hat{u}(k, t) = \hat{f}(k) e^{-k^2 t}.$$

$$\hat{u}(k, 0) = \hat{f}(k)$$

$$\therefore u(x, t) = \left(\hat{f}(k) e^{-k^2 t} \right)^{\vee}$$

$$= \frac{1}{2\pi} f * (e^{-k^2 t})^{\vee}.$$

Compute $(e^{-k^2 t})^{\vee}$. We showed earlier in the notes:

$$(2\pi)^{-\frac{1}{2}} \sigma \left(e^{-\sigma^2 k^2 / 2} \right)^{\vee} = e^{-x^2 / 2\sigma^2}. \quad \text{Let } \sigma = (2t)^{\frac{1}{2}}.$$

$$\Rightarrow (e^{-t k^2})^{\vee} = (2\pi)^{\frac{1}{2}} (2t)^{-\frac{1}{2}} e^{-x^2 / 4t}.$$

$$\therefore u(x, t) = \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(x-u) e^{-u^2 / 4t} du.$$

ex) Solve $i \partial_t u + \partial_x \partial_x u = 0$, with $u(x, 0) = f(x)$, $f \rightarrow 0$ as $x \rightarrow \pm \infty$.

$$i \partial_t \hat{u} + (ik)^2 \hat{u} = 0 \Rightarrow \hat{u}(k, t) = \hat{f}(k) e^{-ik^2 t}.$$

$$\partial_t \hat{u} = -ik^2 \hat{u}$$

$$\therefore u(x, t) = \int_{-\infty}^{\infty} \hat{f}(k) e^{-ik^2 t} e^{ikx} dk.$$