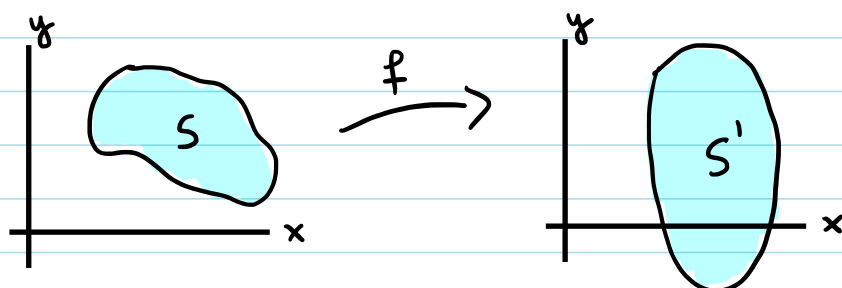


Functions of a complex variable

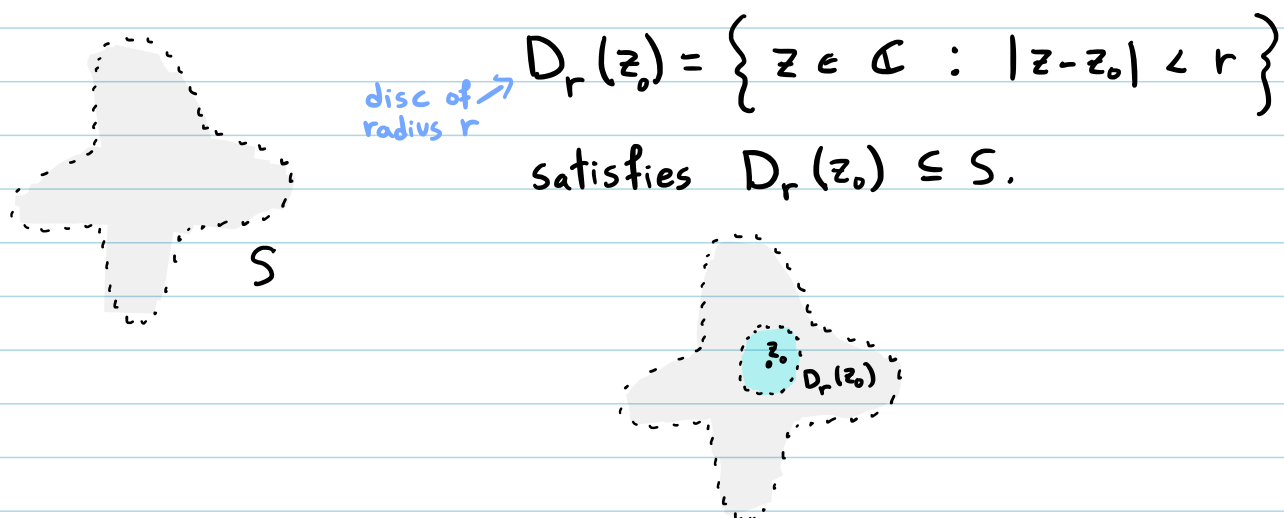
We consider functions of $z \in \mathbb{C}$.



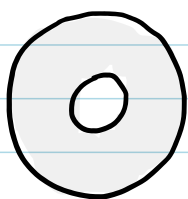
Notation: $f(x, y) = u(x, y) + i v(x, y)$.

Some terminology on sets $S \subseteq \mathbb{C}$:

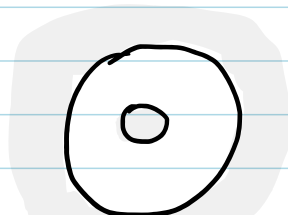
(i) S is open if: $\forall z_0 \in S, \exists r > 0$ s.t.



(ii) S is connected if: every pair of points in S can be joined by a finite number of line segments lying entirely in S .



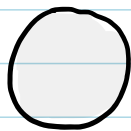
S is connected
 $S = \{z : 1 < |z| < 2\}$



S is not connected
 $S = \{z : |z| < 1 \text{ or } |z| > 2\}$

(iii) A domain $S \subseteq \mathbb{C}$ is an open connected set.

(iv) The boundary of S consists of all points z_0 such that: every disc $D_r(z_0) = \{ |z - z_0| < r \}$ contains at least one point in S and one point not in S .



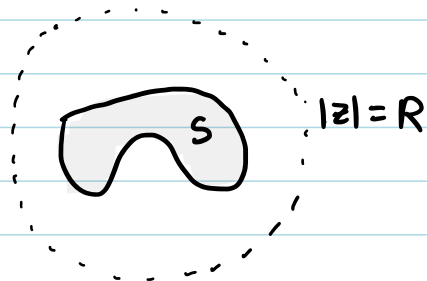
$$S = \{ |z| \leq 1 \}$$

$$\text{bdd}(S) = \{ |z| = 1 \}$$

(v) S is closed if it contains all its boundary.

(vi) S is bounded if there exists $R > 0$ s.t.

$$S \subseteq \{ |z| < R \}.$$



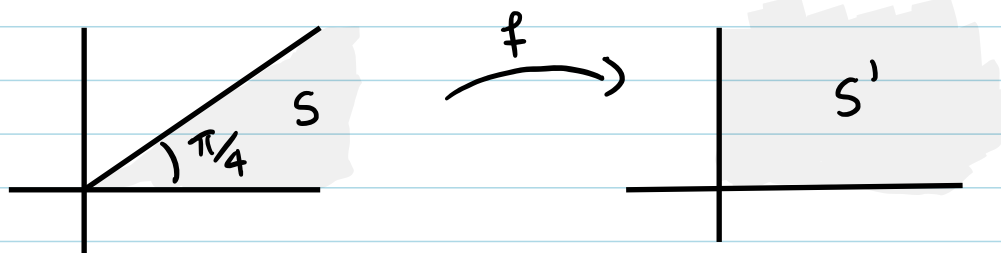
"Find the image" problems

ex) Find the image of $S = \{ z : 0 \leq \text{Arg}(z) \leq \frac{\pi}{4} \}$

under $f(z) = z^2$. $S' = f(S) = ?$

Write $w = f(z)$, $z = r e^{i\varphi}$ with $0 \leq \varphi \leq \pi/4$.

$$w = r^2 e^{2i\varphi}, \quad 0 \leq 2\varphi \leq \pi/2.$$



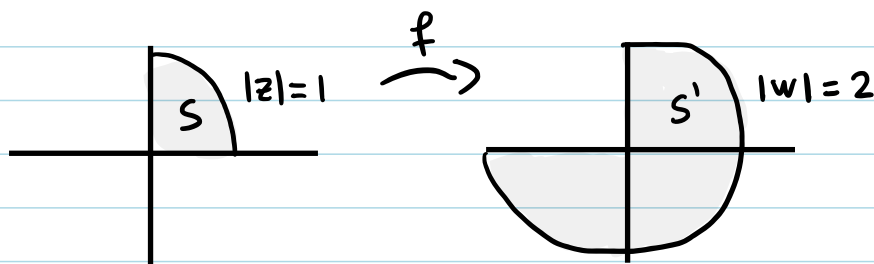
$$S' = \{ w : 0 \leq \text{Arg}(w) \leq \pi/2 \}$$

ex) Find the image of $S = \{ z : |z| \leq 1, 0 \leq \text{Arg}(z) \leq \frac{\pi}{2} \}$
under $w = f(z) = -2z^3$. $S' = f(S) = ?$

Write $z = re^{i\varphi}$, $0 \leq r \leq 1$, $0 \leq \varphi \leq \pi/2$.
 $w = -2r^3 e^{3i\varphi}$

$$w = \underbrace{2r^3}_{\leq 2} \exp(i(\underbrace{\pi}_{\pi \leq} + \underbrace{3\varphi}_{\leq \pi + 3(\frac{\pi}{2})}))$$

$$S' = \{ w : |w| \leq 2, \pi \leq \text{Arg}(w) \leq \frac{5\pi}{2} \}$$



ex) Find the image of $S = \{ z : |z-1| \leq 1 \}$
under $w = f(z) = 1/z$.

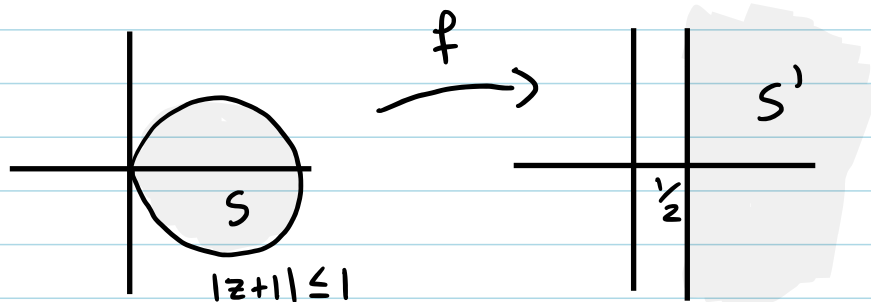
$$\text{Since } z = 1/w \Rightarrow |1/w - 1| \leq 1$$

$$\Rightarrow |w-1| \leq |w|.$$

$$\Rightarrow |w-1|^2 \leq |w|^2$$

$$\begin{aligned} \text{Let } w = u + iv. \Rightarrow (u-1)^2 + v^2 &\leq u^2 + v^2 \\ \Rightarrow -2u + 1 &\leq 0 \\ \Rightarrow u &\geq \frac{1}{2} \end{aligned}$$

$$S' = \{ w : \operatorname{Re}(w) \geq \frac{1}{2} \} = f(S).$$

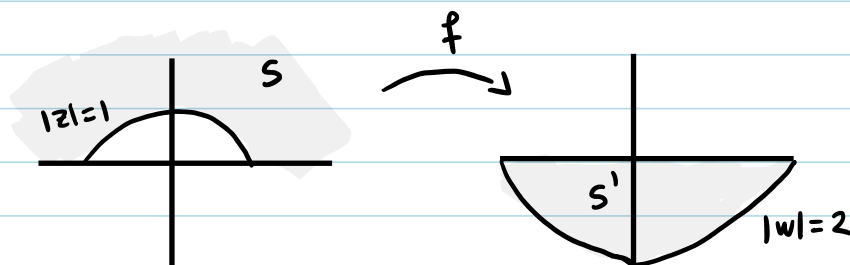


ex) Find the image of $S = \{ z : |z| \geq 1, \operatorname{Im}(z) \geq 0 \}$ under $w = f(z) = \frac{2}{z}$.

Write $z = r e^{i\varphi}$, $r \geq 1$, $0 \leq \varphi \leq \pi$.

$$w = \frac{2}{r} e^{-i\varphi} = \underbrace{\rho}_{\rho \leq 2} e^{i\theta} \quad \begin{matrix} \swarrow \\ -\pi \leq \theta \leq 0 \end{matrix}$$

$$S' = \{ w : |w| \leq 2, \operatorname{Im}(w) \leq 0 \}$$



Note: • $w = z + \beta$

translation

• $w = e^{i\theta_0} z$

rotation by angle θ_0

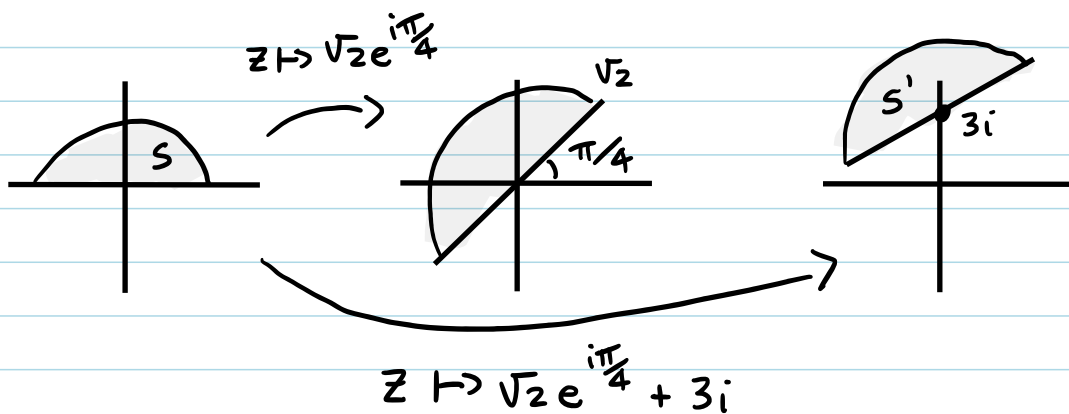
• $w = az$, $a > 0$

stretching if $a > 1$
contraction if $a < 1$

ex) Find the image of $S = \{ z : |z| \leq 1, \operatorname{Im}(z) \geq 0 \}$

under $w = f(z) = (1+i)z + 3i$.

$$(1+i) = \sqrt{2} e^{i\pi/4} \Rightarrow w - 3i = \sqrt{2} e^{i\pi/4} z.$$



$$S' = \left\{ w : |w - 3i| \leq \sqrt{2}, \frac{\pi}{4} \leq \operatorname{Arg}(w - 3i) \leq \frac{5\pi}{4} \right\}$$

Analytic Functions

Def: $f: S \rightarrow \mathbb{C}$ is analytic on a domain $S \subseteq \mathbb{C}$ if:

(A) $f(x,y) = u(x,y) + i v(x,y)$ where the partial derivatives of u and v exist and are continuous.

(B) The Cauchy-Riemann equations hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

ex) $f(x,y) = -2xy + i(x^2 - y^2)$

$$\begin{aligned} u &= -2xy \Rightarrow u_x = -2y & v_x &= 2x \Rightarrow u_x = v_y \\ v &= x^2 - y^2 \Rightarrow u_y = -2x & v_y &= -2y \Rightarrow u_y = -v_x \end{aligned}$$

$\therefore f$ is analytic over $\mathbb{C}$.

ex) $f(x,y) = x$

$u = x \Rightarrow u_x = 1 \neq 0 = v_y \Rightarrow f$ is not analytic.
 $v = 0$

Intuition for analytic functions: they can be written just using $z = x + iy$.

ex) $f = z$ ✓ analytic

ex) $f = \bar{z} + i$ ✗ not analytic

ex) $f = z\bar{z}$ ✗ not analytic

Change of variables: $\begin{cases} z = x + iy \\ \bar{z} = x - iy \end{cases} \quad \begin{cases} x = (\frac{1}{2})(z + \bar{z}) \\ y = (-i/2)(z - \bar{z}) \end{cases}$

Can write $f(x,y)$ or $f(z, \bar{z})$ for a general function $f: \mathbb{C} \rightarrow \mathbb{C}$.

Chain rule: $\frac{\partial}{\partial \bar{z}} f = \frac{\partial x}{\partial \bar{z}} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial \bar{z}} \frac{\partial f}{\partial y}$

$$= \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right)$$

$\therefore \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$

claim: Cauchy-Riemann eqn $\Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0$.
 $f = u + iv, \quad u_x = v_y$
 $u_y = -v_x$

check: $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (u(x,y) + i v(x,y))$

$$= \frac{1}{2} (u_x - v_y + i(v_x + u_y)). \quad \checkmark$$

Summary: Let f be C^1 on $S \subseteq \mathbb{C}$. Then:

$$f \text{ analytic on } S \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0 \quad \text{"no } \bar{z} \text{ appearing"}$$

Note: " f is C^1 " means: $f = u(x,y) + iv(x,y)$ and partials of u and v exist and are continuous.

Notation: If f is analytic, write $f'(z) := \frac{\partial f}{\partial z}$.

Terminology: "entire" means analytic on all $\mathbb{C}$.

ex) $f(z) = z^2 \Rightarrow f'(z) = 2z$

Properties:

(a) $(f + cg)' = f' + cg'$ f, g analytic, $c \in \mathbb{C}$.

(b) $(fg)' = f'g + gf'$

(c) $\frac{d}{dz} f(g(z)) = f'(g(z)) g'(z)$

(d) $\frac{d}{dz} z^n = n z^{n-1}$ $n = \pm 1, \pm 2, \dots$

ex) Show $f(z) = e^z$ is analytic and $f'(z) = f(z)$.

$$f = e^x \cos y + i e^x \sin y$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (e^x \cos y + i e^x \sin y)$$

$$= \frac{1}{2} (e^x \cos y + i e^x \sin y - i e^x \sin y - e^x \cos y) = 0.$$

Since f is C^1 and $\frac{\partial f}{\partial \bar{z}} = 0$, f is analytic.

$$\begin{aligned}
 \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (e^x \cos y + i e^x \sin y) \\
 &= \frac{1}{2} (e^x \cos y + i e^x \sin y + i e^x \sin y + e^x \cos y) \\
 &= e^z. \quad \Rightarrow f'(z) = e^z.
 \end{aligned}$$

ex) $f(z) = 1/z$ is analytic on $\mathbb{C} \setminus 0$. $\frac{\partial}{\partial \bar{z}} \left(\frac{1}{z} \right) = 0$.

In terms of x, y : $\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}$.

$\therefore f(x, y) = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}$ solves Cauchy-Riemann eqn.

Theorem: If f is analytic in a domain $S \subseteq \mathbb{C}$ and $f'(z) = 0$ everywhere in S , then $f \equiv \text{const}$ in S .

Proof: Let $f(x, y) = u(x, y) + i v(x, y)$.

$$0 = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (u + i v)$$

$$0 = (u_x + v_y) + i (v_x - u_y).$$

$$\begin{aligned}
 \therefore \begin{cases} u_x = -v_y \\ u_y = v_x \end{cases} \quad \text{Combine this with Cauchy-Riemann:} \\
 \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \begin{matrix} v_y = -v_y \\ v_x = -v_x \end{matrix} &\Rightarrow \begin{matrix} v_x = 0 \\ v_y = 0 \end{matrix} &\Rightarrow \begin{matrix} u_x = 0 \\ u_y = 0 \end{matrix}
 \end{aligned}$$

Since $\nabla v = 0$, v is constant.
 $\nabla u = 0$, u is constant.

$\therefore f$ is constant.

□

Harmonic Functions

Let $u: S \rightarrow \mathbb{R}$ be a real-valued function on a domain $S \subseteq \mathbb{R}^2$.

Suppose all 2nd order partials of u exist.

Say u is harmonic if $\nabla^2 u = 0$.

Recall: $\nabla^2 u = \partial_x \partial_x u + \partial_y \partial_y u$.

Theorem: If $f = u + iv$ is analytic on $S \subseteq \mathbb{C}$, and 2nd partials of u, v are continuous, then $u(x, y)$ and $v(x, y)$ are harmonic on $S \subseteq \mathbb{R}^2$.

Proof: $u_{xx} + u_{yy} = \partial_x u_x + \partial_y u_y$

$$= \partial_x v_y - \partial_y v_x \quad \text{Cauchy-Riemann}$$

$$= v_{xy} - v_{yx} \quad \text{Order of partials}$$

$$= 0.$$

□ $v_{xx} + v_{yy} = 0$ is similar

ex) $f = e^z$

$$u(x, y) = e^x \cos y \quad \nabla^2 u = 0$$

$$v(x, y) = e^x \sin y \quad \nabla^2 v = 0.$$

ex) Find an analytic function f s.t. $\operatorname{Re}(f) = xy - x + y$
 $f(0) = 0$.

$$f = u + iv, \quad u = xy - x + y.$$

Cauchy-Riemann

$$u_x = v_y \Rightarrow y - 1 = v_y \quad (1)$$

$$u_y = -v_x \Rightarrow x + 1 = -v_x \quad (2)$$

$$(1) \Rightarrow v(x, y) = \frac{y^2}{2} - y + g(x). \quad (3)$$

$$\textcircled{2} \text{ and } \textcircled{3} \Rightarrow x+1 = -\frac{\partial}{\partial x} \left(\frac{y^2}{2} - y + g(x) \right)$$

$$x+1 = -g'(x)$$

$$\Rightarrow g(x) = -\frac{x^2}{2} - x + C.$$

$$\text{Sub into } \textcircled{3} \Rightarrow v(x,y) = -\frac{x^2}{2} + \frac{y^2}{2} - x - y + C.$$

$$f = xy - x + y + i \left(-\frac{x^2}{2} + \frac{y^2}{2} - x - y \right), \quad \text{since } C=0 \text{ because } f(0)=0.$$

ex) $f(z) = z^5$ is analytic.

$$\Rightarrow \operatorname{Re}(z^5) = \operatorname{Re}(r^5 e^{i5\theta})$$

$$= r^5 \cos 5\theta \quad \text{is harmonic.}$$

ex) Similarly: $\nabla^2 (r^n \cos n\theta) = 0$

$$\nabla^2 (r^n \sin n\theta) = 0.$$