

Elementary Functions

Recall the exponential map $e^z: \mathbb{C} \rightarrow \mathbb{C}$.

$$e^z = e^x e^{iy} = e^x (\cos y + i \sin y), \quad z = x + iy.$$

Recall: $e^{z_1} = e^{z_2} \Leftrightarrow z_1 = z_2 + i(2\pi k), \quad k = 0, \pm 1, \pm 2, \dots$

ex) Solve $e^z = -4i$.

$$\text{Write } -4i = e^{\log 4} e^{-i\pi/2}.$$

$$\text{So } e^z = e^{\log 4} e^{-i\pi/2} \Rightarrow z = \log 4 - i\frac{\pi}{2} + i(2\pi k) \\ k = 0, \pm 1, \pm 2, \dots$$

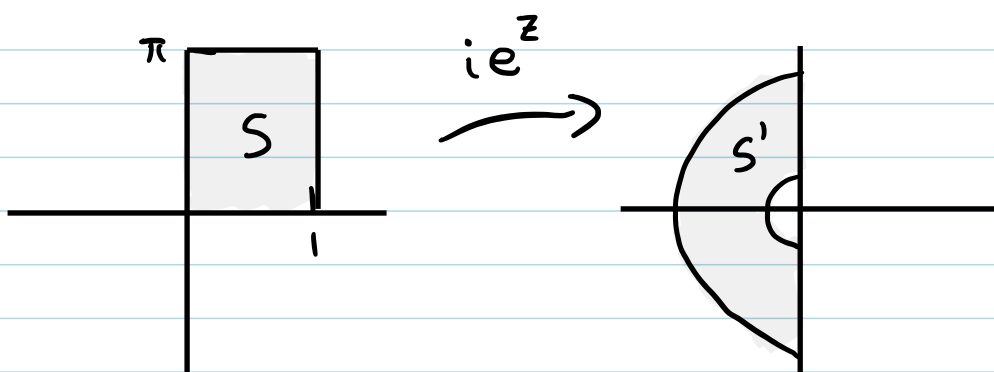
ex) Find the image of $S = \left\{ z : \begin{array}{l} 0 \leq \operatorname{Re} z \leq 1, \\ 0 \leq \operatorname{Im} z \leq \pi \end{array} \right\}$
under $w = i e^z$.

$$w = e^{i\pi/2} e^x e^{iy}, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq \pi.$$

$$w = \rho e^{i\theta}, \quad \rho = e^x, \quad 1 \leq \rho \leq e^1$$

$$\theta = \frac{\pi}{2} + y, \quad \frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} + \pi.$$

$$S' = \{ w = e^{iz} : z \in S \} = \left\{ w : \begin{array}{l} 1 \leq |w| \leq e \\ \operatorname{Re}(w) \leq 0 \end{array} \right\}$$

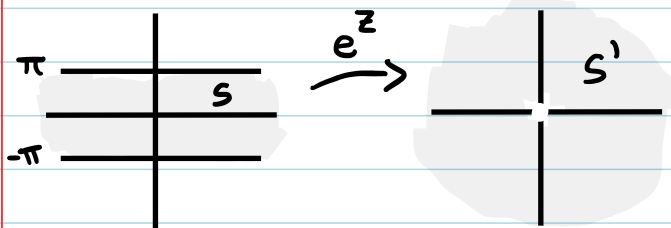


ex) Find the image of $S = \{z : -\pi \leq \operatorname{Im} z \leq \pi\}$ under $w = e^z$.

$$S' = \{w = e^z : z \in S\} = ?$$

$$w = e^x e^{iy} = \rho e^{i\theta}, \quad \begin{aligned} \rho = e^x &\Rightarrow 0 < \rho < \infty. \\ \theta = y &\Rightarrow -\pi \leq \theta \leq \pi. \end{aligned}$$

$$S' = \mathbb{C} \setminus \{0\}.$$



Sines and Cosines

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Note: Since e^{iz} and e^{-iz} are analytic, we see that $\sin z$ and $\cos z$ are also analytic.

$$\begin{aligned} \text{Note: for } x \in \mathbb{R}, \quad & \frac{1}{2i} (e^{ix} - e^{-ix}) \\ &= \frac{1}{2i} (\cos x + i \sin x - \cos x - i \sin(-x)) \\ &= \sin x. \text{ Consistent.} \end{aligned}$$

Properties:

$$(i) \frac{d}{dz} \sin z = \cos z, \quad \frac{d}{dz} \cos z = -\sin z$$

$$(ii) \sin(z + 2\pi) = \sin z, \quad \cos(z + 2\pi) = \cos z.$$

$$(iii) \sin(-z) = -\sin z, \quad \cos(-z) = \cos z$$

$$(iv) \sin^2 z + \cos^2 z = 1$$

$$(v) \overline{\sin z} = \sin \bar{z}, \quad \overline{\cos z} = \cos \bar{z}.$$

Hyperbolic functions

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

$$\cosh z = \frac{e^z + e^{-z}}{2}.$$

Real and imaginary parts of sine and cosine

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$

check cos identity:

$$\cos z = \frac{1}{2} (e^{iz} + e^{-iz})$$

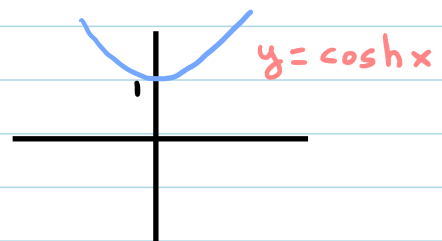
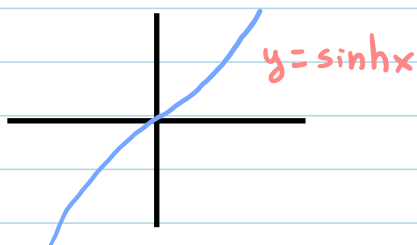
$$= \frac{1}{2} e^{-y} e^{ix} + \frac{1}{2} e^y e^{-ix}$$

$$= \frac{1}{2} e^{-y} (\cos x + i \sin x) + \frac{1}{2} e^y (\cos x - i \sin x)$$

$$= \cos x \left(\frac{1}{2} (e^y + e^{-y}) \right) - i \sin x \left(\frac{e^y - e^{-y}}{2} \right).$$

Recall:

graphs of
real hyperbolic



ex) Solve $\sin z = \cosh(4)$.

Decompose into real + imaginary:

$$\sin z = \cosh y \sin x + i \cos x \sinh y = \cosh(4).$$

$$\Rightarrow \cos x \sinh y = 0.$$

$$\Rightarrow y=0 \text{ or } x = \frac{\pi}{2} + k\pi, \quad k=0, \pm 1, \pm 2, \dots$$

But if $y=0$, then $\cosh(0) \sin x = \cosh(4)$. No solutions when $y=0$.
" ≤ 1 > 1

$$\text{So: } x = \frac{\pi}{2} + k\pi.$$

$$\Rightarrow \cosh y \sin\left(\frac{\pi}{2} + k\pi\right) = \cosh(4) \Rightarrow \text{need } k=\text{even so that } \sin\left(\frac{\pi}{2} + k\pi\right) = +1.$$

>0 ± 1 >0

$$\Rightarrow \cosh y = \cosh(4)$$

$$\Rightarrow y=4 \text{ or } y=-4.$$

$$\therefore z = \frac{\pi}{2} + 2\pi k \pm 4i, \quad k=0, \pm 1, \pm 2, \dots$$

ex) Solve $\cos(z) = 100$.

$$\frac{e^{iz} + e^{-iz}}{2} = 100 \Rightarrow e^{iz} + e^{-iz} - 200 = 0.$$

$$\Rightarrow (e^{iz})^2 - 200(e^{iz}) + 1 = 0.$$

$$\therefore e^{iz} = \frac{200 \pm \sqrt{200^2 - 4}}{2}$$

$$= 100 \pm (100^2 - 1)^{1/2}$$

$$= 100 \pm \sqrt{9999} > 0.$$

$$e^{iz} = e^{\ln(100 \pm \sqrt{9999})}$$

$$\Rightarrow iz = \ln(100 \pm \sqrt{9999}) + i(2\pi k), \quad k=0, \pm 1, \pm 2, \dots$$

$$\Rightarrow z = 2\pi k - i \ln(100 \pm \sqrt{9999}).$$

Note: no real solutions since $|\cos x| \leq 1$ for $x \in \mathbb{R}$.

Note: $|\cos(iy)| = \cosh y \geq 1$ for $y \in \mathbb{R}$.

Principal branch of the logarithm

For $z = r e^{i\varphi}$, $-\pi < \varphi \leq \pi$, define

$$\text{Log } z = \ln r + i\varphi.$$

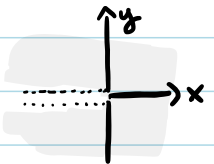
Note: Capital "L" is used to distinguish the principal Log from the multi-valued logarithm $\log z$ with a lower case "l" which will be used later.

Note: $\text{Log } z$ is the "principal" branch of the logarithm because $\varphi = \text{Arg}(z)$.

↖ principal argument

Thm: $f(z) = \text{Log } z$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$.

Pf: Rewrite in (x, y) -coords:



$$f(x, y) = \ln(x^2 + y^2)^{\frac{1}{2}} + i\varphi(x, y).$$

① Compute $\frac{\partial \varphi}{\partial x}$, $\frac{\partial \varphi}{\partial y}$.

$$\begin{aligned} x &= r \cos \varphi & y &= r \sin \varphi \\ \Rightarrow \text{differentiate in } x & \quad 1 = \partial_x r \cos \varphi - r \sin \varphi \partial_x \varphi \\ & \quad 0 = \partial_x r \sin \varphi + r \cos \varphi \partial_x \varphi \end{aligned}$$

$$\begin{aligned} \Rightarrow \sin \varphi &= \partial_x r \sin \varphi \cos \varphi - r \sin^2 \varphi \partial_x \varphi & (\sin \varphi) \\ 0 &= \partial_x r \sin \varphi \cos \varphi + r \cos^2 \varphi \partial_x \varphi & (\cos \varphi) \end{aligned}$$

subtract eqn

$$\Rightarrow \sin \varphi = -r (\sin^2 \varphi + \cos^2 \varphi) \partial_x \varphi.$$

$$\Rightarrow \partial_x \varphi = -\frac{\sin \varphi}{r} = \frac{-y}{x^2 + y^2}.$$

$$\text{Similarly, } \partial_y \varphi = \frac{x}{x^2 + y^2}. \quad \text{omitted.}$$

② Compute $\frac{\partial}{\partial \bar{z}}$

$$\begin{aligned}\frac{\partial}{\partial \bar{z}} f &= \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \left(\frac{1}{2} \ln(x^2 + y^2) + i \varphi(x, y) \right) \\ &= \frac{1}{2} \left(\frac{x}{x^2 + y^2} - \underbrace{i \frac{y}{x^2 + y^2}}_{i \partial_x \varphi} + \underbrace{i \frac{y}{x^2 + y^2}}_{- \partial_y \varphi} - \frac{x}{x^2 + y^2} \right) = 0 \quad \checkmark\end{aligned}$$

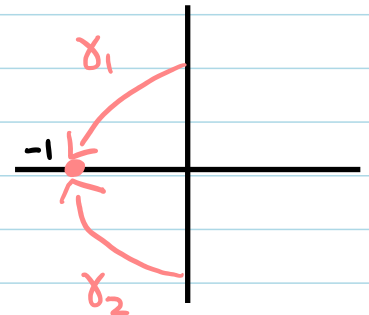
$\therefore \frac{\partial f}{\partial \bar{z}} = 0$ + f is C^1 on $\mathbb{C} \setminus (-\infty, 0]$ $\Rightarrow f$ is analytic. on $\mathbb{C} \setminus (-\infty, 0]$ $\square$

Note: Although $\text{Log } z$ is defined on $\mathbb{C} \setminus \{0\}$, it is not even continuous across the negative real axis.

Let $\gamma_1(t) = e^{i\pi t}, 0 \leq t \leq 1$

$\gamma_2(t) = e^{-i\pi t}, 0 \leq t \leq 1$

$\lim_{t \rightarrow 1} \gamma_1(t) = -1 = \lim_{t \rightarrow 1} \gamma_2(t)$



but $\lim_{t \rightarrow 1} \text{Log } \gamma_1(t) = i\pi$

$\lim_{t \rightarrow 1} \text{Log } \gamma_2(t) = -i\pi$

$\text{Log}(-1) = i\pi$ but when approaching along $\gamma_2(t)$ we see the Log become $-i\pi$.

Derivative of $\text{Log } z$

$f(z) = \text{Log } z \Rightarrow f' = \frac{1}{z}$
on $\mathbb{C} \setminus (-\infty, 0]$

Proof: Write in (x, y) coords:

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \left(\frac{1}{2} \ln(x^2 + y^2) + i \varphi(x, y) \right)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{x}{x^2+y^2} + i \frac{\partial \varphi}{\partial x} - i \frac{y}{x^2+y^2} + \frac{\partial \varphi}{\partial y} \right) \quad \text{Computed } \partial_x \varphi, \partial_y \varphi \text{ earlier}$$

$$= \frac{1}{2} \left(\frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2} - \frac{iy}{x^2+y^2} + \frac{x}{x^2+y^2} \right)$$

$$= \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2}$$

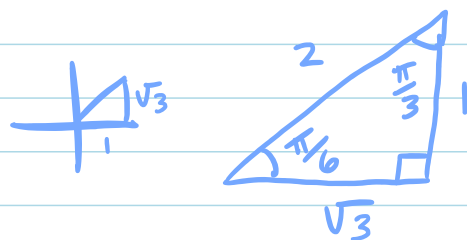
$$\Rightarrow \frac{\partial f}{\partial \bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{1}{z} \quad \square$$

ex) Calculate $\text{Log}(-i)$.

$$\text{Log}(-i) = \text{Log} e^{-i\pi/2} = -i \frac{\pi}{2}$$

ex) Calculate $\text{Log}(1 + i\sqrt{3})$.

$$\begin{aligned} \text{Log}(1 + i\sqrt{3}) &= \text{Log}(2e^{i\pi/3}) \\ &= \ln 2 + i \frac{\pi}{3} \end{aligned}$$



Properties of $\text{Log} z$

(i) $\text{Log}(z_1 z_2) \neq \text{Log} z_1 + \text{Log} z_2$ ✗

(ii) $\text{Log} e^z \neq z$ ✗ unless $-\pi < \text{Im}(z) \leq \pi$

(iii) $z = e^{\text{Log} z}$ ✓

(i) $\text{Log}(e^{i\pi} e^{i\pi/2}) = \text{Log} e^{-i\pi/2} = -i\pi/2$
 $\text{Log}(e^{i\pi}) + \text{Log}(e^{i\pi/2}) = i\pi + i\pi/2$

(ii) $\text{Log} e^{i(3\pi/2)} = -i\frac{\pi}{2}$

(iii) $e^{\text{Log} z} = e^{\ln|z|} e^{i\varphi} = z$

ex) Find the image of $S = \{z : \operatorname{Im} z \geq 0\}$ under $w = f(z) = 2 \operatorname{Log} z$. $S' = \{w : w = f(z), z \in S\} = ?$

Write $z = r e^{i\varphi}$, $0 \leq \varphi \leq \pi$.

$$\Rightarrow 2 \operatorname{Log} z = 2 \ln r + i (2\varphi)$$

$$\Rightarrow w = a + i b \quad \text{with} \quad -\infty < a < \infty \\ 0 \leq b \leq 2\pi$$

$$S' = \{w : 0 \leq \operatorname{Im} w \leq 2\pi\}$$

