

Multi-Valued Functions

Given z , what are the solutions w to $e^w = z$?

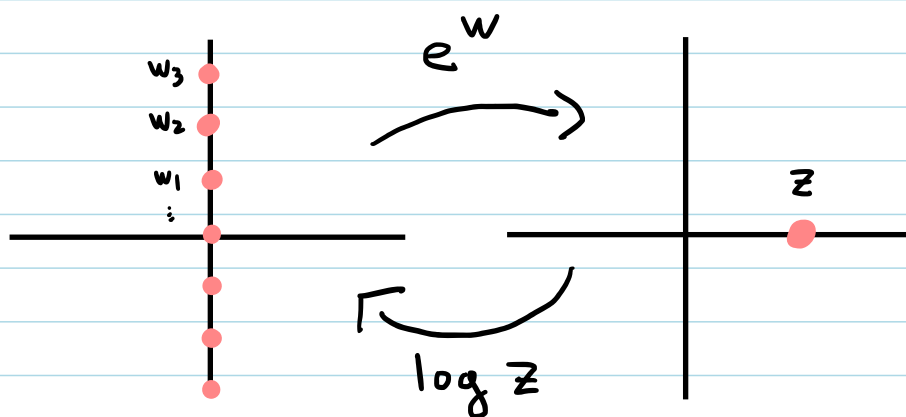
$$w = \log z := \ln |z| + i(\operatorname{Arg}(z) + 2\pi k), \quad k = 0, \pm 1, \pm 2, \dots$$

where $z = |z|e^{i\varphi}$, $-\pi < \varphi \leq \pi$, $\operatorname{Arg}(z) = \varphi$.

We call $\log z$ the multi-valued logarithm.

Recall: $\operatorname{Log} z = \ln |z| + i \operatorname{Arg}(z)$ was introduced earlier. It is one choice among the many values of $\log z$.

$\operatorname{Log} z$ = "principal branch of $\log$ "



ex) $\log(2i) = ?$

$$2i = 2e^{i\pi/2} \Rightarrow \log 2i = \ln 2 + i\left(\frac{\pi}{2} + 2\pi k\right), \quad k = 0, \pm 1, \pm 2, \dots$$

Some Properties:

(i) $\log(z_1 z_2) = \log z_1 + \log z_2$

(ii) $\log\left(\frac{1}{z}\right) = -\log z$

(iii) $\log z^n \neq n \log z$.

$$\log 1^2 = i(2\pi k) \quad k = 0, \pm 1, \pm 2, \dots$$

$$2\log 1 = i(4\pi k) \quad k = 0, \pm 1, \pm 2, \dots$$

Exponents

Let $\alpha \in \mathbb{C}, \alpha \neq 0$. Define the multi-valued function:

$$z^\alpha = e^{\alpha \log z}.$$

$$\text{i.e. } z^\alpha = e^{\alpha (\ln|z| + i(\text{Arg } z + 2\pi k))} \quad k=0, \pm 1, \pm 2, \dots$$

ex) Compute i^i .

$$i^i = e^{i \log i}, \quad \log i = \ln 1 + i \frac{\pi}{2} + i(2\pi k), \quad k=0, \pm 1, \pm 2, \dots$$

$$\Rightarrow i^i = e^{-\frac{\pi}{2}} e^{-2\pi k}, \quad k=0, \pm 1, \pm 2, \dots$$

$$\text{ex) } 9^{1/2} = e^{\frac{1}{2}(\ln 9 + i 2\pi k)} \\ = \pm 3.$$

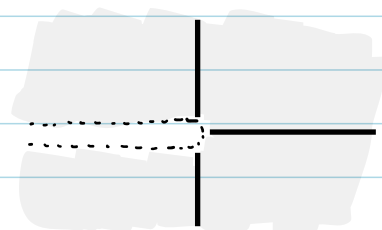
Def: The principal value of z^α is:

$$z^\alpha = e^{\alpha \text{Log } z}.$$

ex) Compute the principal value of i^i .

$$i^i = e^{i \text{Log } i} = e^{i(i\frac{\pi}{2})} = e^{-\pi/2}.$$

Note: Since $\text{Log } z$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$,
then z^α is analytic on $\mathbb{C} \setminus (-\infty, 0]$.



$$\text{Also: } \frac{d}{dz} z^\alpha = \alpha z^{\alpha-1} \text{ on } \mathbb{C} \setminus (-\infty, 0].$$

ex) Let $f(z) = \text{Log} \left(\frac{z-1}{z-2} \right)$. Where is $f(z)$ analytic?

$\text{Log } w$ is analytic off the bad set $\begin{cases} \text{Im } w = 0 \\ \text{Re } w \leq 0 \end{cases}$.

$\Rightarrow \text{Log} \left(\frac{z-1}{z-2} \right)$ is analytic off the bad set:

$$\text{Im} \left(\frac{z-1}{z-2} \right) = 0, \quad \text{Re} \left(\frac{z-1}{z-2} \right) \leq 0.$$

$$\frac{z-1}{z-2} \cdot \frac{\bar{z}-2}{\bar{z}-2} = \frac{|z|^2 - 2z - \bar{z} + 2}{|z-2|^2}$$

$$= \frac{x^2 + y^2 - 3x + 2 - iy}{|z-2|^2}.$$

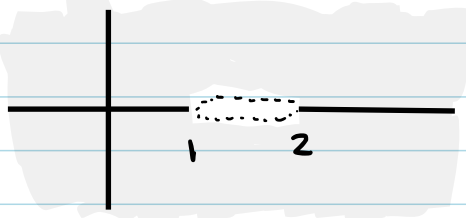
$$\text{Im} \left(\frac{z-1}{z-2} \right) = 0 \Rightarrow y = 0.$$

$$\text{Re} \left(\frac{z-1}{z-2} \right) \leq 0 \Rightarrow x^2 - 3x + 2 \leq 0 \\ (x-1)(x-2) \leq 0.$$



$\therefore f$ is analytic on $\mathbb{C} \setminus S$,

$$S = \{x + iy : y = 0, 1 \leq x \leq 2\}$$



Square roots

There are two values of $z^{\frac{1}{2}}$.

$$z^{\frac{1}{2}} = e^{\frac{1}{2} \log z} = e^{\frac{1}{2}(\ln|z| + \text{Arg } z + 2\pi i k)} \quad k = 0, \pm 1, \pm 2, \dots$$

$$= \left\{ \underbrace{e^{\frac{1}{2}(\ln|z| + \text{Arg } z)}}_{\text{principal branch}}, -e^{\frac{1}{2}(\ln|z| + \text{Arg } z)} \right\}$$

$$\therefore z^{\frac{1}{2}} = \pm |z|^{\frac{1}{2}} e^{i \text{Arg}(z)/2} \quad \text{multi-valued square root}$$

ex) $4^{\frac{1}{2}} = \pm 2.$

Main property: $(z^{\frac{1}{2}})^2 = z.$

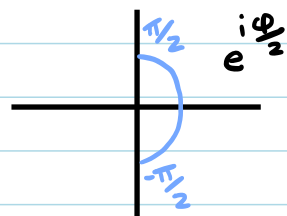
ex) $az^2 + bz + c = 0$ is solved by

$$z = \frac{-b \pm (b^2 - 4ac)^{\frac{1}{2}}}{2a} \quad \leftarrow \text{multivalued}$$

Principal Branch: $\sqrt{z} = |z|^{\frac{1}{2}} e^{i \text{Arg}(z)/2}.$

Note: $\text{Re } \sqrt{z} \geq 0$ on principal branch, since:

$$\sqrt{z} = |z|^{\frac{1}{2}} e^{i\varphi/2}, \quad -\pi < \varphi \leq \pi.$$



ex) Compute the principal value of:
 $(1+i)^{\frac{1}{2}}.$

$$1+i = \sqrt{2} e^{i\pi/4} \Rightarrow \sqrt{1+i} = 2^{\frac{1}{4}} e^{i\pi/8}.$$

ex) $(-1)^{\frac{1}{2}} = \pm i$ multi-valued

ex) $\sqrt{-1} = i$ on principal branch.

ex) Find solutions to: $z^{\frac{1}{2}} + 2 - i = 0$
 where $z^{\frac{1}{2}}$ denotes the principal value.

$$\operatorname{Re} z^{\frac{1}{2}} = -2 < 0 \Rightarrow \text{no solutions.}$$

Wrong branch for this eqn!

Consider $z = 3 - 4i$.

Then

$$\begin{aligned} (-2+i)^2 &= 3-4i \\ z &= (-2+i)^2 \Rightarrow \sqrt[alt]{z} + 2 - i = 0. \end{aligned}$$

Alternate square root: $z = r e^{i\varphi}$, $-\pi < \varphi \leq \pi$.

$$\sqrt[alt]{z} = -r^{\frac{1}{2}} e^{i\varphi/2}$$

e.g. $\sqrt[alt]{4} = -2$.

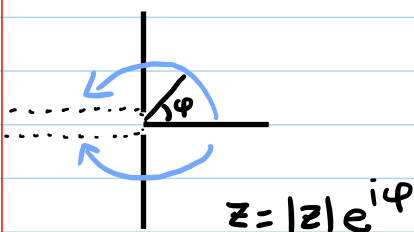
We call $f(z) = \sqrt[alt]{z}$ a different choice of branch of $f(z) = z^{\frac{1}{2}}$.

High school: Introduce imaginary numbers to take square roots of negative numbers.

$$\sqrt{-1} = i.$$

Complex analysis: Let $f(z) = \sqrt{z}$ be the principal branch. Then f is analytic on $\mathbb{C} \setminus (-\infty, 0]$.

⚠ $\sqrt{-1}$ is in "bad set" $(-\infty, 0]$.
 $\sqrt{z}$ jumps at -1 .



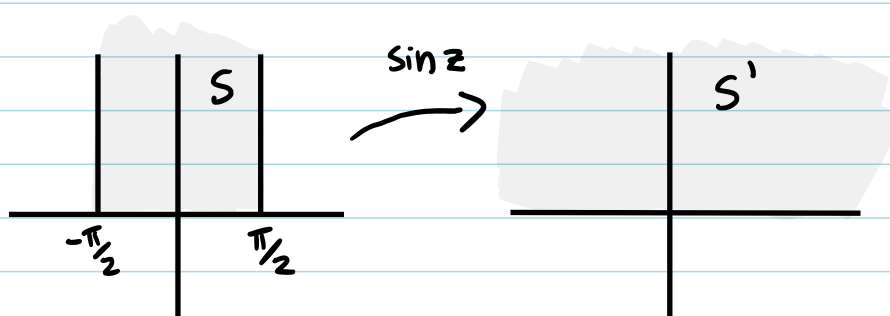
$$\begin{aligned} \sqrt{z} &\xrightarrow{\varphi \rightarrow \pi} |z|^{\frac{1}{2}} e^{i\pi/2} = i |z|^{\frac{1}{2}} \\ \sqrt{z} &\xrightarrow{\varphi \rightarrow -\pi} |z|^{\frac{1}{2}} e^{-i\pi/2} = -i |z|^{\frac{1}{2}} \end{aligned}$$

Inverting Sine

Proposition: Let $S = \{z : |\operatorname{Re}(z)| \leq \pi/2, \operatorname{Im}(z) \geq 0\}$

The image of S under $f(z) = \sin z$ is:

$$S' = \{w : \operatorname{Im}(w) \geq 0\}.$$



Proof: First, note $f(S) \subseteq S'$. This is because:

$$\sin(x+iy) = \sin x \cosh y + i \underbrace{\cos x}_{\geq 0} \underbrace{\sinh y}_{\geq 0}.$$

$-\pi/2 \leq x \leq \pi/2 \quad y \geq 0$

Main step: show that for any w with $\operatorname{Im} w \geq 0$, we can find $z \in S$ so that $f(z) = w$.

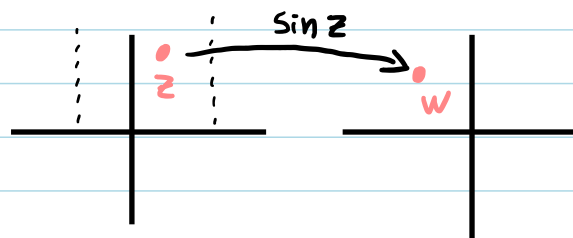
① Use quadratic formula

$$\sin z = w \Leftrightarrow \frac{1}{2i} (e^{iz} - e^{-iz}) = w$$

$$\Leftrightarrow (e^{iz})^2 - 2iw e^{iz} - 1 = 0.$$

Quadratic formula:
$$e^{iz} = \frac{2iw \pm \sqrt{-4w^2 + 4}}{2}.$$

Guess: Keep $e^{iz} = iw + \sqrt{1-w^2}$, with principal square root.



② Guess: keep principal log: select this soln to $\sin z = w$.

$$z = -i \operatorname{Log} (i w + \sqrt{1-w^2})$$

let's write $z^{\frac{1}{2}} = \sqrt{z}$ for princ branch in this proof.

$$z = \operatorname{Arg} (i w + (1-w^2)^{\frac{1}{2}}) - i \ln |i w + (1-w^2)^{\frac{1}{2}}|.$$

We found z , but we need $z \in S$. This means:

③

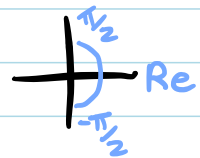
$$\bullet |\operatorname{Re}(z)| \leq \pi/2 \Leftrightarrow -\frac{\pi}{2} \leq \operatorname{Arg} (i w + (1-w^2)^{\frac{1}{2}}) \leq \frac{\pi}{2}$$

④

$$\bullet \operatorname{Im}(z) \geq 0 \Leftrightarrow |i w + (1-w^2)^{\frac{1}{2}}| < 1.$$

Proof of ③: We need to show

$$\operatorname{Re}(i w + (1-w^2)^{\frac{1}{2}}) \geq 0.$$



This is equivalent to: $\operatorname{Re} (1-w^2)^{\frac{1}{2}} \stackrel{(*)}{\geq} -\operatorname{Re}(i w) = \operatorname{Im} w$,

Use identity: $\operatorname{Re} \sqrt{z} \stackrel{(**)}{=} \sqrt{\frac{1}{2} (|z| + \operatorname{Re} z)}$. proof: later

$$\Rightarrow \operatorname{Re} (1-w^2)^{\frac{1}{2}} = \left(\frac{1}{2} (|1-w^2| + \operatorname{Re}(1-w^2)) \right)^{\frac{1}{2}}.$$

So, $(*)$ squared becomes:

$$\frac{1}{2} (|1-w^2| + 1 - \operatorname{Re} w^2) \geq (\operatorname{Im} w)^2$$

$$\Leftrightarrow \frac{1}{2} (|1-w^2| + 1 - |w|^2) \geq 0.$$

True by reverse triangle ineq
 $|1-w^2| \geq |1-|w|^2|$.

$$\begin{aligned} w &= a+ib \\ \operatorname{Re} w^2 &= a^2-b^2 \\ |w|^2 &= a^2+b^2 \\ \operatorname{Re} w^2 + 2(\operatorname{Im} w)^2 &= |w|^2 \end{aligned}$$

This proves ③ once we understand $(**)$.

Proof of identity (**)

$$\text{Let } z = x + iy, \quad (u + iv)^2 = x + iy \Rightarrow \begin{aligned} & \overset{(a)}{u^2 - v^2} = x \\ & 2iuv = iy \end{aligned}$$

$$\Rightarrow |z| \overset{(b)}{=} u^2 + v^2$$

$$\Rightarrow \underbrace{u^2 - v^2}_{(a)} + \underbrace{u^2 + v^2}_{(b)} = x + |z| \Rightarrow 2u^2 = x + |z| \Rightarrow u = \left(\frac{x + |z|}{2} \right)^{\frac{1}{2}} \checkmark$$

Proof of (4): $|iw + (1 - w^2)^{\frac{1}{2}}| < 1$.

$$\text{Note: } (iw + (1 - w^2)^{\frac{1}{2}}) \frac{(-iw + (1 - w^2)^{\frac{1}{2}})}{(-iw + (1 - w^2)^{\frac{1}{2}})} = \frac{w^2 + 1 - w^2}{(-iw + (1 - w^2)^{\frac{1}{2}})}$$

$$\Rightarrow |iw + (1 - w^2)^{\frac{1}{2}}| = \frac{1}{|-iw + (1 - w^2)^{\frac{1}{2}}|}$$

$$\text{We show: } |-iw + (1 - w^2)^{\frac{1}{2}}| \overset{(*)}{\geq} 1. \text{ This proves (4).}$$

$$\begin{aligned} |-iw + (1 - w^2)^{\frac{1}{2}}|^2 &= (-iw + (1 - w^2)^{\frac{1}{2}}) \overline{(-iw + (1 - w^2)^{\frac{1}{2}})} \\ &= |w|^2 + |1 - w^2|^2 + i \underbrace{\left(\bar{w} (1 - w^2)^{\frac{1}{2}} - w \overline{(1 - w^2)^{\frac{1}{2}}} \right)}_{\text{claim: } \geq 0} \end{aligned}$$

Assuming claim:

$$|-iw + (1 - w^2)^{\frac{1}{2}}|^2 \geq |w|^2 + |1 - w^2|^2$$

$$\geq 1 \quad \overset{(*)}{}$$

$$\begin{aligned} & \text{by triangle ineq:} \\ & 1 = |1 - w^2 + w^2| \\ & \leq |1 - w^2| + |w^2|. \end{aligned}$$

So the final step is:

$$(**) = i \left(\bar{w} (1 - w^2)^{\frac{1}{2}} - w \overline{(1 - w^2)^{\frac{1}{2}}} \right) \geq 0, \quad \text{provided } \text{Im } w \geq 0.$$

Expand using: $w = a + ib$, $b \geq 0$
 $(1-w^2)^{\frac{1}{2}} = \alpha + i\beta$ $\alpha \geq 0$. $\operatorname{Re} \sqrt{\dots} \geq 0$

$$(**) = i(a-ib)(\alpha+i\beta) - i(a+ib)(\alpha-i\beta)$$

$$= -2\alpha\beta + 2b\alpha.$$

$$\underbrace{-2\alpha\beta}_{\geq 0}$$

Will show: $\alpha\beta \leq 0$.

$$\begin{aligned} \text{Expand: } (\alpha+i\beta)^2 &= 1-w^2 \\ &= 1-(a+ib)^2 \end{aligned}$$

$$\text{Take Im part: } 2\alpha\beta i = -2abi$$

$$ab\beta = -\alpha\beta^2 < 0 \Rightarrow \alpha\beta \leq 0. \quad \checkmark$$

$$b \geq 0$$

$$\therefore (**) \geq 0. \quad \square$$

Branch Chasing

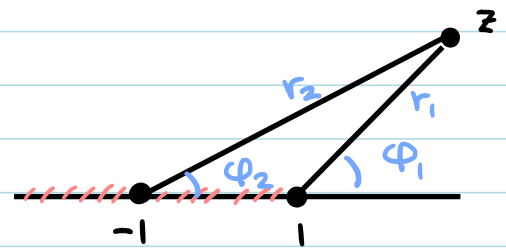
ex) Construct a branch of $f(z) = (z^2 - 1)^{1/2}$ that is analytic on $\mathbb{C} \setminus [-1, 1]$ and s.t. $f(2) = \sqrt{3}$. Find $f(-i)$.

$$\begin{aligned} \text{Write } f(z) &= (z-1)^{1/2} (z+1)^{1/2} \\ &= (r_1 e^{i\varphi_1})^{1/2} (r_2 e^{i\varphi_2})^{1/2} \end{aligned}$$

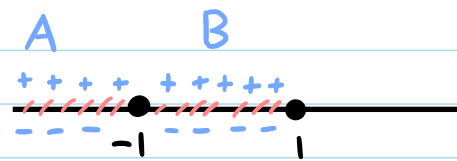
$$\therefore f(z) = r_1^{1/2} r_2^{1/2} e^{i\varphi_1/2} e^{i\varphi_2/2}$$

Choose range of angles:

$$-\pi < \varphi_1 \leq \pi, \quad -\pi < \varphi_2 \leq \pi.$$



Check continuity across interfaces:



$$\begin{aligned} \text{Interface A: } \varphi_1^+ &= \pi & \varphi_2^+ &= \pi \\ \varphi_1^- &= -\pi & \varphi_2^- &= -\pi \end{aligned}$$

$$e^{i(\varphi_1 + \varphi_2)^+/2} = e^{i\pi} \quad \text{Continuous across interface A.}$$

$$e^{i(\varphi_1 + \varphi_2)^-/2} = e^{-i\pi} \quad "$$

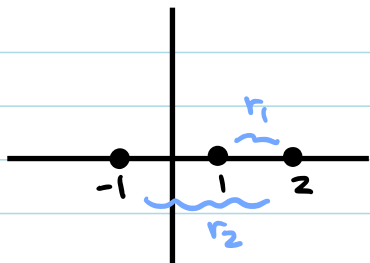
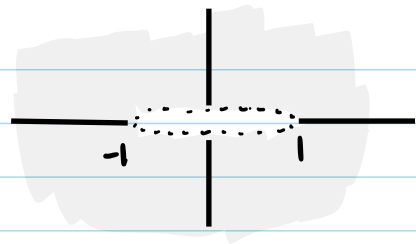
$$\begin{aligned} \text{Interface B: } \varphi_1^+ &= \pi & \varphi_2^+ &= 0 \\ \varphi_1^- &= -\pi & \varphi_2^- &= 0 \end{aligned}$$

$$e^{i(\varphi_1 + \varphi_2)^+/2} = e^{i\pi/2} \quad \text{discontinuous across interface B.}$$

$$e^{i(\varphi_1 + \varphi_2)^-/2} = e^{-i\pi/2} \quad \neq$$

As constructed, $f(z)$ is analytic on:

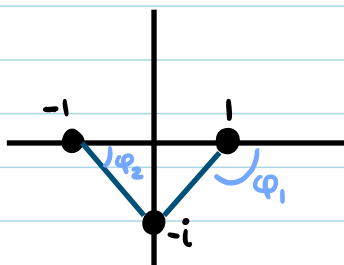
Compute $f(z)$:



$$\begin{aligned}\varphi_1 &= 0 & r_1 &= 1 \\ \varphi_2 &= 0 & r_2 &= 3\end{aligned}$$

$$f(z) = (1 \cdot 3)^{\frac{1}{2}} e^{0/2} = \sqrt{3}$$

Compute $f(-i)$:



$$\begin{aligned}\varphi_1 &= -3\pi/4 & r_1 &= \sqrt{2} \\ \varphi_2 &= -\pi/4 & r_2 &= \sqrt{2}\end{aligned}$$

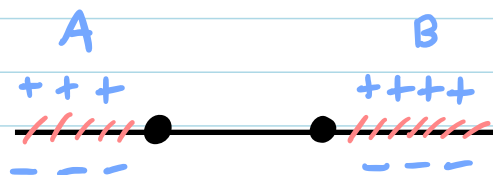
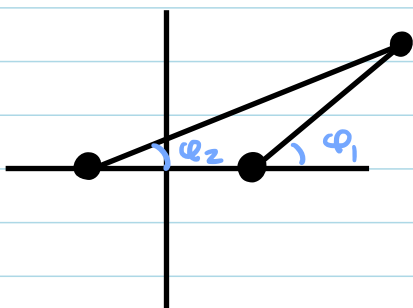
$$f(-i) = (\sqrt{2}\sqrt{2})^{\frac{1}{2}} e^{i(-\frac{3\pi}{4} - \frac{\pi}{4})/2}$$

$$= \sqrt{2} e^{-i\pi/2} = -i\sqrt{2}.$$

ex) Construct a branch of $f(z) = (z^2 - 1)^{\frac{1}{2}}$ that is analytic on $\mathbb{C} \setminus (-\infty, -1] \cup [1, \infty)$ and s.t. $f(0) = i$. Compute $f(-i)$.

$$f(z) = (r_1 r_2)^{\frac{1}{2}} e^{i(\varphi_1 + \varphi_2)/2},$$

$$\begin{aligned}0 &\leq \varphi_1 < 2\pi \\ -\pi &< \varphi_2 \leq \pi.\end{aligned}$$



Interface A:

$$\begin{aligned}\varphi_1^+ &= \pi \\ \varphi_1^- &= \pi \\ \varphi_2^+ &= \pi \\ \varphi_2^- &= -\pi\end{aligned}$$

$$e^{i(\varphi_1^+ + \varphi_2^+)/2} = e^{i\pi}$$

$$e^{i(\varphi_1^- + \varphi_2^-)/2} = e^0 \quad \text{discontinuous!}$$

Interface B:

$$\varphi_1^+ = 0$$

$$\varphi_1^- = 2\pi$$

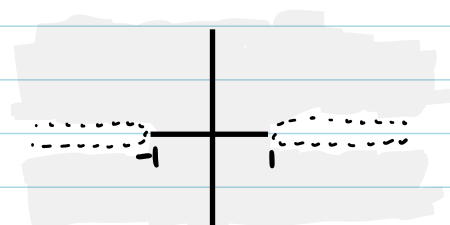
$$\varphi_2^+ = 0$$

$$\varphi_2^- = 0$$

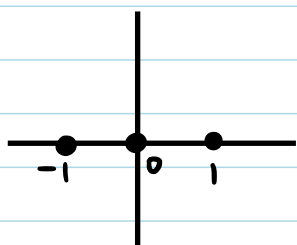
$$e^{i(\varphi_1^+ + \varphi_2^+)/2} = e^{i\pi}$$

$$e^{i(\varphi_1^- + \varphi_2^-)/2} = e^0 \quad \text{discontinuous!}$$

$\therefore f(z)$ is analytic on:



Compute $f(0)$:



$$\varphi_1 = \pi$$

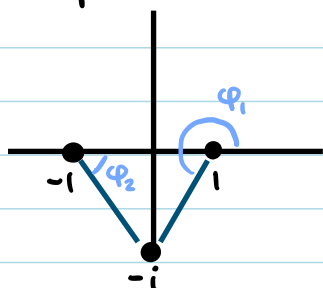
$$\varphi_2 = 0$$

$$r_1 = 1$$

$$r_2 = 1$$

$$f(0) = (1 \cdot 1)^{1/2} e^{i\pi/2} = i$$

Compute $f(-i)$:



$$\varphi_1 = 5\pi/4$$

$$\varphi_2 = -\pi/4$$

$$r_1 = \sqrt{2}$$

$$r_2 = \sqrt{2}$$

$$f(-i) = (\sqrt{2} \sqrt{2})^{1/2} e^{i\pi/2} = \sqrt{2} i$$

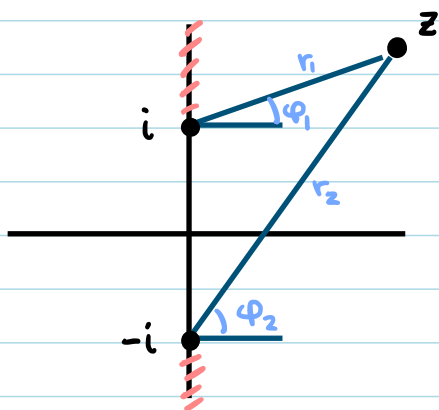
ex) Construct a branch of $f(z) = (z^2 + 1)^{1/3}$ analytic on: s.t. $f(0) = \exp\left(\frac{2\pi i}{3}\right)$.



compute $f(i/2)$.

$$f(z) = (z-i)^{1/3} (z+i)^{1/3}$$

$$f(z) = (r_1 r_2)^{1/3} e^{i(\varphi_1 + \varphi_2)/3}$$



Try: $-\frac{3\pi}{2} < \varphi_1 \leq \frac{\pi}{2}$



$-\frac{\pi}{2} < \varphi_2 \leq \frac{3\pi}{2}$



Good news: evidently continuous on $\mathbb{C} \setminus i(-\infty, -1] \cup i[1, \infty)$ so no interfaces to check.

Bad news: $f(0) = 1 e^{i0} = 1$.
Wrong branch.

Try $f(z) = (r_1 r_2)^{\frac{1}{3}} e^{i(\tilde{\varphi}_1 + \varphi_2)/3}$, $\tilde{\varphi}_1 = \varphi_1 + 2\pi$,
 $\frac{\pi}{2} < \tilde{\varphi}_1 \leq \frac{5\pi}{2}$.

Compute $f(0)$: $f(0) = 1 e^{i(-\frac{\pi}{2} + 2\pi + \frac{\pi}{2})/3} = e^{i2\pi/3}$ ✓

Compute $f(i/2)$: $f(i/2) = \left(\frac{1}{2} \frac{3}{2}\right)^{\frac{1}{3}} e^{i2\pi/3} = \left(\frac{3}{4}\right)^{\frac{1}{3}} e^{i2\pi/3}$.

