

Contour Integrals

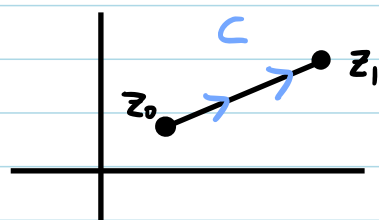
Def: Let C be a directed curve parametrized by

$z(t)$ on $a \leq t \leq b$. Then

$$\int_C f(z) dz := \int_a^b f(z(t)) \frac{dz}{dt} dt.$$

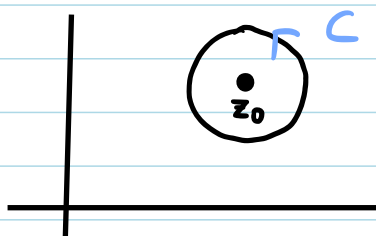
ex) Parametrize C = straight line from z_0 to z_1 .

$$z(t) = z_0(1-t) + t z_1, \\ 0 \leq t \leq 1.$$



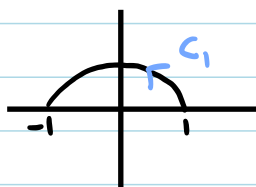
ex) Parametrize C = circle centred at z_0 of radius r , going counter-clockwise.

$$z(t) = z_0 + r e^{it} \\ -\pi \leq t \leq \pi$$



ex) $\int_{C_1} \bar{z} dz = ?$ C_1 = Semi-circle C_2 = Semi-circle

$$\int_{C_2} \bar{z} dz = ?$$



Calculate C_1 :

$$z(t) = e^{it} \quad \frac{dz}{dt} = i e^{it} \\ 0 \leq t \leq \pi$$

$$\int_{C_1} \bar{z} dz = \int_0^\pi e^{-it} (i e^{it}) dt = i \int_0^\pi dt = i\pi.$$

Calculate C_2 :

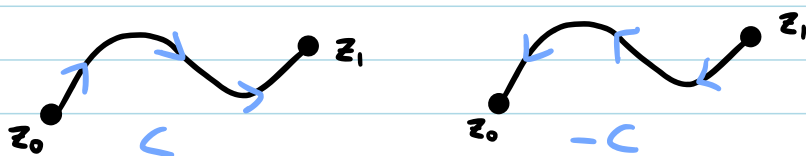
$$z(t) = e^{it}$$

$$0 \leq t \leq -\pi$$

$$\int_{C_2} \bar{z} dz = \int_0^{-\pi} e^{-it} (ie^{it}) dt = i \int_0^{-\pi} dt = -i\pi.$$

Notation: C = curve from z_0 to z_1 ,

$-C$ = curve in reverse from z_1 to z_0 .

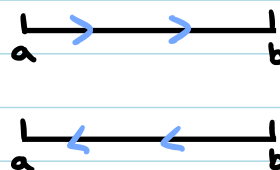


Identity:

$$\int_C f dz = - \int_{-C} f dz$$

Analogy from Calc I:

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$



Length of a curve

$$\text{Length}(C) = \int_a^b \left| \frac{dz}{dt} \right| dt.$$

$$\text{This is because: } \left| \frac{dz}{dt} \right| = \left(\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right)^{1/2}$$

$$z(t) = x(t) + iy(t).$$

$$\left| \frac{dz}{dt} \right| = \text{speed of } z(t).$$

Basic Integral Estimate

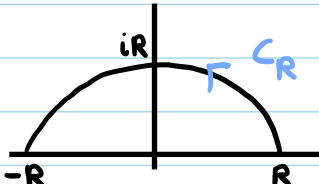
$$\left| \int_C f dz \right| \leq \left(\max_C |f(z)| \right) (\text{Length}(C))$$

Proof:

$$\left| \int_a^b f(z) \frac{dz}{dt} dt \right| \leq \int_a^b |f(z(t))| \left| \frac{dz}{dt} \right| dt$$

$$\left(\left| \int_a^b f(z) \frac{dz}{dt} dt \right| \leq \int_a^b |f(z(t))| \left| \frac{dz}{dt} \right| dt \right)$$

$$\leq \left(\max_C |f(z)| \right) \underbrace{\int_a^b \left| \frac{dz}{dt} \right| dt}_{\text{length}(C)} \quad \square$$

ex) Compute $\lim_{R \rightarrow \infty} \int_{C_R} \frac{z}{z^3+1} dz$, $C_R =$ 

Reverse triangle ineq:

$$|z^3 - 1| \geq ||z|^3 - 1|. \Rightarrow |z^3 + 1| \geq R^3 - 1 \text{ on } C_R. \quad R > 1000$$

$$\Rightarrow \frac{|z|}{|z^3+1|} \leq \frac{R}{R^3-1} \text{ on } C_R.$$

$$\Rightarrow \left| \int_{C_R} \frac{z}{z^3+1} dz \right| \leq \frac{R}{R^3-1} \underbrace{\text{Length}(C_R)}_{= \pi R}$$

$$= \frac{\pi R^2}{R^3-1} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

$$\therefore \lim_{R \rightarrow \infty} \int_{C_R} \frac{z}{z^3+1} dz = 0.$$

Fundamental Theorem of Calculus

Suppose $f(z)$ is continuous in a domain $D \subseteq \mathbb{C}$ and there exists $F(z)$ in D such that: $F' = f$.

$$\text{Then: } \int_C f dz = F(z_f) - F(z_i)$$

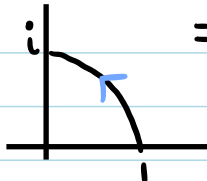
for all curves C in D travelling from z_i to z_f .

Recall: $F'(z)$ means: $\frac{\partial F}{\partial \bar{z}} = 0$ and $F' := \frac{\partial F}{\partial z}$. $F \in C^1(D)$

Proof: Parametrize C by $z(t)$, $t_i \leq t \leq t_f$.

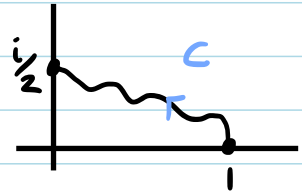
$$\begin{aligned} \int_{t_i}^{t_f} f(z(t)) \frac{dz}{dt} dt &= \int_{t_i}^{t_f} \frac{d}{dt} F(z(t)) dt & \frac{d}{dt} F(z(t)) &= \underbrace{\frac{\partial F}{\partial z} \frac{dz}{dt}}_{\substack{= f \\ \text{by}}} + \underbrace{\frac{\partial F}{\partial \bar{z}} \frac{d\bar{z}}{dt}}_{=0} \\ &= \left|_{t_i}^{t_f} F \right. & \text{FTC} \\ &= F(t_f) - F(t_i). \quad \square \end{aligned}$$

ex) Calculate:

$$\int_C e^z dz \quad \text{where } C = i \text{ to } 1 = \text{quarter unit circle from } 1 \text{ to } i.$$


Notice $F(z) = e^z$ solves $F' = e^z$.

$$\int_C e^z dz = F(i) - F(1) = e^i - e = \cos(1) - e + i \sin(1).$$

ex) Calculate $\int_C z^3 dz$ where $C =$ 

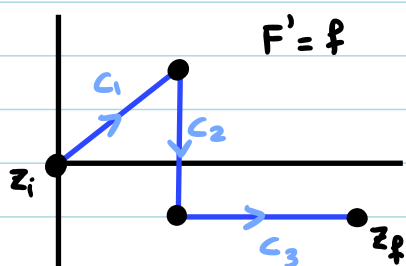
Notice: $F(z) = \frac{z^4}{4}$ solves $F' = z^3$.

$$\begin{aligned} \int_C z^3 dz &= F(i/2) - F(1) \\ &= \frac{1}{4} \left(\frac{i^4}{2^4} - 1^4 \right) = \frac{1}{4} \left(\frac{1-2^4}{2^4} \right) = \frac{-15}{64}. \end{aligned}$$

Note: If C is broken into pieces $C_1, \dots, C_n$, then

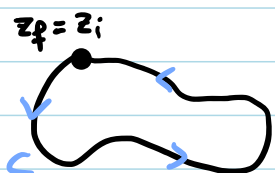
$$\int_C f dz := \sum_{k=1}^n \int_{C_k} f dz, \text{ and}$$

$$\int_C f dz = F(z_f) - F(z_i) \text{ still holds.}$$



ex) Compute $\int_C (e^z + 1) dz$ for any loop C .

$$\begin{aligned} F(z) &= e^z + z \\ F'(z) &= e^z + 1 \end{aligned} \Rightarrow \int_C (e^z + 1) dz = F(z_f) - F(z_i) = 0.$$



$$\int_C f(z) dz = 0 \text{ if } F' = f.$$

ex) Compute $\int_C \bar{z} dz$ where C is the unit circle clockwise.

Note: cannot solve $F'(z) = \bar{z}$. $\frac{\partial F}{\partial \bar{z}} = 0$ and $\frac{\partial F}{\partial z} = \bar{z}$.

Parametrize $-C$ by: $z(t) = e^{it}$, $-\pi \leq t \leq \pi$.

Flip sign of final answer.

$$\int_{-C} \bar{z} dz = \int_{-\pi}^{\pi} (\overline{e^{it}})(ie^{it}) dt = i \int_{-\pi}^{\pi} dt = 2\pi i.$$

$$\Rightarrow \int_C \bar{z} dz = -2\pi i \neq 0.$$



The most important contour integral

$$\int_C (z - z_0)^n dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n = 1, \pm 2, \pm 3, \pm 4, \dots \end{cases}$$

where C = circle of radius R centred at z_0
going counter-clockwise.

Parametrize C by:

$$z(t) = z_0 + Re^{it}$$

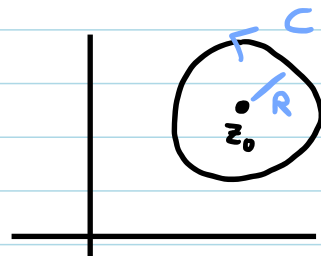
$$0 \leq t \leq 2\pi$$

$$\int_C (z - z_0)^n dz = \int_0^{2\pi} (Re^{it})^n (iRe^{it}) dt$$

$$= R^{n+1} \int_0^{2\pi} i e^{i(n+1)t} dt.$$

Case 1: $n = 1, \pm 2, \pm 3, \dots$

$$= R^{n+1} \int_0^{2\pi} (n+1)^{-1} e^{i(n+1)t} = 0. \quad e^{i(n+1)(2\pi)} = e^0 = 1$$

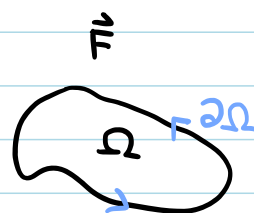


Case 2: $n = -1$

$$\begin{aligned} \int_C (z - z_0)^{-1} dz &= R^{-1+1} \int_0^{2\pi} i e^{i(-1+1)t} dt \\ &= \int_0^{2\pi} i dt = 2\pi i. \end{aligned}$$

Review of Green's Theorem

$$\int_{\partial\Omega} \vec{F} \cdot d\vec{r} = \iint_{\Omega} (\text{2D-curl } \vec{F}) dA,$$



where: $\Omega \subseteq \mathbb{R}^2$ domain with boundary $\partial\Omega$
: Boundary $\partial\Omega$ has arrows with Ω "on your left"
: $\vec{F} = (F_1(x,y), F_2(x,y))$ is C^1 .

$$\text{2D-curl } \vec{F} = \begin{vmatrix} \partial_x & \partial_y \\ F_1 & F_2 \end{vmatrix} = \partial_x F_2 - \partial_y F_1.$$

Alternate version:

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

$$\begin{aligned} \vec{F} &= (P, Q) \\ d\vec{r} &= \left(\frac{dx}{dt}, \frac{dy}{dt} \right) dt \end{aligned}$$

Complex version:

$$\int_{\partial\Omega} f(z) dz = 2i \iint_{\Omega} \frac{\partial f}{\partial \bar{z}} dA$$

$$dA = dx dy$$

Derivation: $f = u + iv$.

$$\int_{\partial\Omega} f dz = \int_{\partial\Omega} (u + iv)(dx + i dy)$$

$$= \int_{\partial\Omega} u dx - v dy + i \int_{\partial\Omega} u dy + v dx.$$

$$= \int_{\partial\Omega} (u, -v) \cdot d\vec{r} + i \int_{\partial\Omega} (v, u) \cdot d\vec{r}$$

$$= \iint_{\Omega} (-\partial_x v - \partial_y u) + i(\partial_x u - \partial_y v) \, dA \quad \text{Green's}$$

$$= \iint_{\Omega} i(\partial_x + i\partial_y)(u + iv) \, dA$$

$$= \iint_{\Omega} \frac{\partial f}{\partial \bar{z}} (z i \, dA) . \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$