

Cauchy's Integral Formula

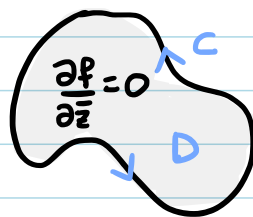
Previously: $\int_C f dz$ for general $f(z, \bar{z})$.

Next: $\int_C f dz$ for analytic $f(z)$.

Thm: Suppose $f(z)$ is analytic inside and on a loop C .

Then: $\int_C f dz = 0$.

Proof: $\int_C f dz = 2i \iint_D \underbrace{\frac{\partial f}{\partial \bar{z}}}_{=0} dA = 0$. $\square$
Green's Thm



Recall: f analytic on $D \subseteq \mathbb{C}$ means: $f \in C^1(D)$ and $\frac{\partial f}{\partial \bar{z}} = 0$.

Technical Remark:

Goursat's Thm, which we won't prove, states:

$$\left(\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} \text{ exists } \forall z \in D \right) \Leftrightarrow \left(f \in C^1(D) \text{ and } \frac{\partial f}{\partial \bar{z}} = 0 \right)$$

both common definitions
of " f is analytic"

ex) Calculate $\int_C \frac{\sin z}{z + 100} dz$, $C = \text{unit circle CCW} =$

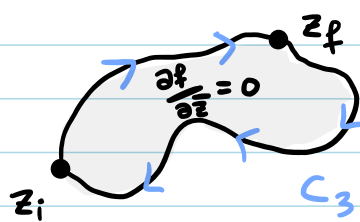
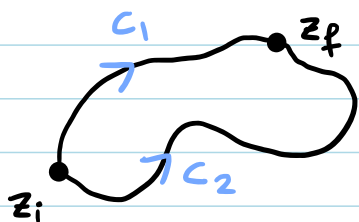
Since $\frac{\sin z}{z + 100}$ is analytic inside C , $\int_C \frac{\sin z}{z + 100} dz = 0$.

Thm: Suppose f is analytic in a disc D .

Then given $z_i, z_f \in D$,

$\int_{C_1} f dz = \int_{C_2} f dz$ for all curves C_1, C_2 from z_i to z_f .
contained in D

Proof: Consider $C_3 = C_1 - C_2$.

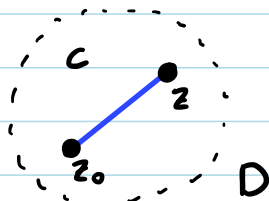


$$0 = \int_{C_3} f dz = \int_{C_1 - C_2} f dz = \int_{C_1} f dz - \int_{C_2} f dz. \quad \square$$

Thm: Suppose $f(z)$ is analytic in a disc D .
Then there exists $F(z)$ such that

$$F'(z) = f(z).$$

Proof: Consider $F(z) = \int_C f(w) dw$ where: C = straight line from z_0 to z .



: $z_0 \in D$ is a chosen fixed reference point.

$$F(z) = \int_0^1 f((1-t)z_0 + tz) (z - z_0) dt. \quad \begin{aligned} z(t) &= (1-t)z_0 + tz \\ \frac{dz}{dt} &= z - z_0. \end{aligned}$$

$f \in C^1(D)$

$$\frac{\partial F}{\partial \bar{z}} = \int_0^1 \frac{\partial}{\partial \bar{z}} \left(f(\gamma(t)) (z - z_0) \right) dt, \quad \text{where: } \gamma(t) = (1-t)z_0 + tz.$$

$$= \int_0^1 (z - z_0) \frac{\partial f(\gamma(t))}{\partial \bar{z}} dt + \int_0^1 f(\gamma(t)) dt$$

$$= \int_0^1 t \frac{d}{dt} f(\gamma(t)) dt + \int_0^1 f(\gamma(t)) dt \quad \begin{aligned} \frac{d}{dt} f(\gamma(t)) &= \frac{\partial f}{\partial z} \frac{dz}{dt} + \frac{\partial f}{\partial \bar{z}} \frac{d\bar{z}}{dt} \\ &= \frac{\partial f}{\partial z} (z - z_0) + \frac{\partial f}{\partial \bar{z}} \cdot 0 \end{aligned}$$

$$= \int_0^1 t f(\gamma(t)) dt - \int_0^1 f(\gamma(t)) dt + \int_0^1 f(\gamma(t)) dt$$

$$\frac{\partial F}{\partial \bar{z}} = f(z) = f(z) \quad \checkmark$$

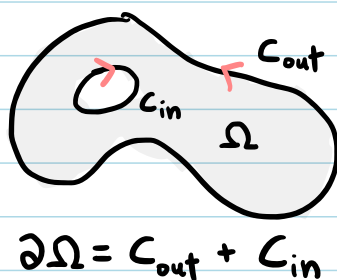
Similar calculation: $\frac{\partial F}{\partial \bar{z}} = 0.$

$$\therefore \frac{\partial F}{\partial \bar{z}} = 0 \quad \text{and} \quad \frac{\partial F}{\partial z} = f. \quad \square$$

Green's Thm with inner circles

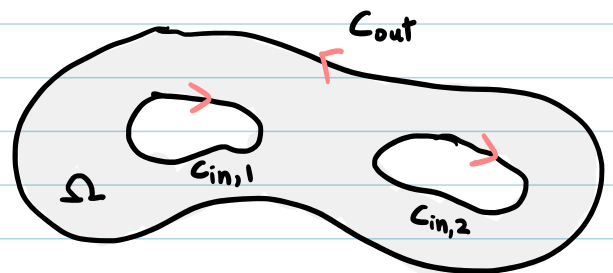
$$\int_{\partial \Omega} f(z) dz = 2i \iint_{\Omega} \frac{\partial f}{\partial \bar{z}} dA$$

ex)



$$\partial \Omega = C_{out} + C_{in}$$

ex)



$$\partial \Omega = C_{out} + C_{in,1} + C_{in,2}$$

"domain on your left"

If f is analytic:

$$2i \iint_{\Omega} \underbrace{\frac{\partial f}{\partial \bar{z}}}_{=0} dA = \int_{C_{out} \text{ CCW}} f dz + \int_{C_{in} \text{ CW}} f dz$$

CW: clockwise
CCW: counterclockwise

$$\int_{C_{out} \text{ CCW}} f dz = - \int_{C_{in} \text{ CW}} f dz.$$

$$\int_{\text{outer boundary CCW}} f dz = \sum \int_{\text{inner boundaries CCW}} f dz.$$

for f analytic on Ω with $\partial \Omega = C_{out} + C_{in}$

ex) Calculate $\int_C \frac{1}{z} dz$ where $C = \text{ellipse } x^2 + 4y^2 = 1$ CCW.

Replace outer ellipse with inner circle C_ϵ .

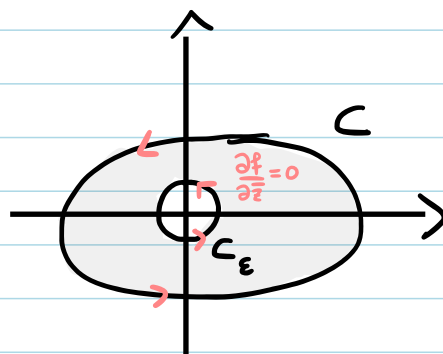
$$\int_C \left(\frac{1}{z}\right) dz = \int_{C_\epsilon} \left(\frac{1}{z}\right) dz.$$

C_ϵ is circle $|z| = \epsilon$ CCW.

C_ϵ is parametrized by:

$$z(t) = \epsilon e^{it}, \quad 0 \leq t \leq 2\pi.$$

$$\begin{aligned} \int_{C_\epsilon} \left(\frac{1}{z}\right) dz &= \int_0^{2\pi} (\epsilon^{-1} e^{-it}) (i\epsilon e^{it}) dt \\ &= 2\pi i. \end{aligned}$$

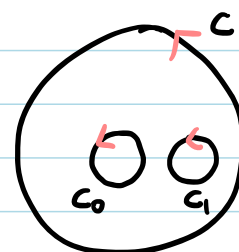


ex) Calculate

$$\int_C \left(\frac{1}{z} + \frac{2}{z-1}\right) dz, \text{ where } C = \text{circle } |z|=10 \text{ CCW.}$$

Replace C with 2 inner circles: $C_0 = |z| = \epsilon$ CCW
 $C_1 = |z-1| = \epsilon$ CCW

$$\begin{aligned} \int_{C_0} \left(\frac{1}{z} + \frac{2}{z-1}\right) dz \\ = \int_0^{2\pi} \epsilon^{-1} e^{-it} (\epsilon i e^{it}) dt + \int_{C_0} \frac{2}{z-1} dz \\ = 2\pi i \quad \quad \quad = 0 \text{ since } \frac{2}{z-1} \text{ is analytic inside } C_0. \end{aligned}$$



$$\begin{aligned} \int_{C_1} \left(\frac{1}{z} + \frac{2}{z-1}\right) dz \\ = \underbrace{\int_{C_1} \left(\frac{1}{z}\right) dz}_{=0} + \underbrace{2 \int_0^{2\pi} \epsilon^{-1} e^{-it} (\epsilon i e^{it}) dt}_{=4\pi i} \end{aligned}$$

$$z(t) = 1 + \epsilon e^{it}$$

$$\int_C = \int_{C_0} + \int_{C_1} = 2\pi i + 4\pi i = 6\pi i.$$

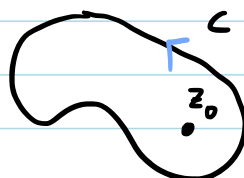
Cauchy Integral Formula

more precise:

C = closed

simple
curve

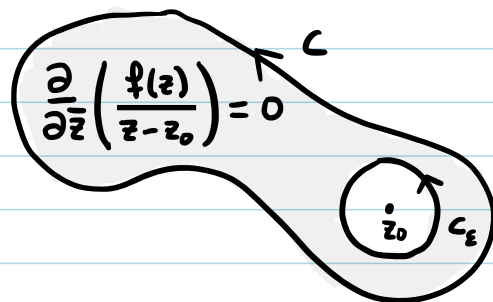
Let C be a loop oriented CCW.
Let $f(z)$ be analytic inside and on C .



$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz, \quad \text{for all } z_0 \text{ inside the loop } C.$$

Proof:

$$\int_C \frac{f(z)}{z - z_0} dz = \int_{|z - z_0| = \varepsilon} \frac{f(z)}{z - z_0} dz$$



by Green's Thm. Let's write $C_\varepsilon = \{ |z - z_0| = \varepsilon \}$.

$$\int_C \frac{f(z)}{z - z_0} dz = \int_{C_\varepsilon} \frac{f(z_0)}{z - z_0} dz + \int_{C_\varepsilon} \frac{f(z) - f(z_0)}{z - z_0} dz$$

$$= f(z_0) \int_0^{2\pi} \varepsilon^{-1} e^{-it} \varepsilon i e^{it} dt + (*)_\varepsilon$$

$$= 2\pi i f(z_0) + (*)_\varepsilon \quad \leftarrow \text{will show this goes to 0.}$$

$$|(*)_\varepsilon| \leq \max_{C_\varepsilon} \left| \frac{f(z) - f(z_0)}{|z - z_0|} \right| \text{Length}(C_\varepsilon)$$

$$= \varepsilon^{-1} \max_{C_\varepsilon} |f(z) - f(z_0)| \cdot 2\pi \varepsilon$$

Since $f(z) \xrightarrow{z \rightarrow z_0} f(z_0)$ by continuity,

$$|(*)_\varepsilon| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

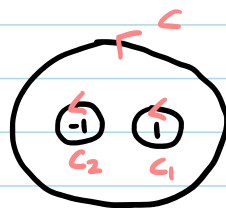
$$\therefore \int_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0).$$

□

ex) Compute $\int_C \frac{dz}{z^2-1}$, $C = \text{circle } |z|=2 \text{ ccw.}$

Note: $\frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)}$ and both $+1, -1$ are inside C .

$$\int_C = \int_{C_1} + \int_{C_{-1}}$$



Near $z=1$:

$$\int_{C_1} \frac{dz}{(z+1)(z-1)} = \int_{C_1} \frac{g(z)}{z-1} dz$$

$C_1: |z-1|=\epsilon \text{ CCW}$

$C_2: |z+1|=\epsilon \text{ CCW}$

$$g(z) = \frac{1}{z+1}, \quad g(1) = \frac{1}{2}.$$

$$\int_{C_1} \frac{dz}{(z+1)(z-1)} = 2\pi i g(1) = \pi i$$

Cauchy integral formula

Near $z=-1$

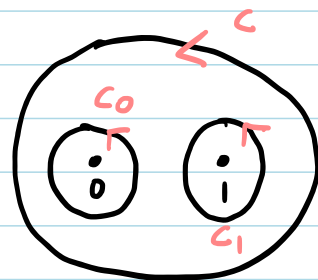
$$\begin{aligned} \int_{C_{-1}} \frac{dz}{(z+1)(z-1)} &= \int_{C_{-1}} \frac{g(z)}{z+1} dz, & g(z) &= \frac{1}{z-1} \\ &= 2\pi i g(-1) & g(-1) &= -\frac{1}{2} \\ &= -\pi i \end{aligned}$$

Add up singular points:

$$\int_C = \pi i - \pi i = 0.$$

ex) $\int_C \frac{3z-2}{z(z-1)} dz, \quad C: |z|=100 \quad \text{ccw}$

Replace C with: $C_0: |z|=\varepsilon \quad \text{ccw}$
 $C_1: |z-1|=\varepsilon \quad \text{ccw}$



Near $z=0$:

$$\begin{aligned} \int_{C_0} \frac{3z-2}{z(z-1)} dz &= \int_{C_0} \frac{g(z)}{z} dz, & g(z) &= \frac{3z-2}{z-1} \\ &= 2\pi i g(0) & g(0) &= 2 \\ &= 4\pi i \end{aligned}$$

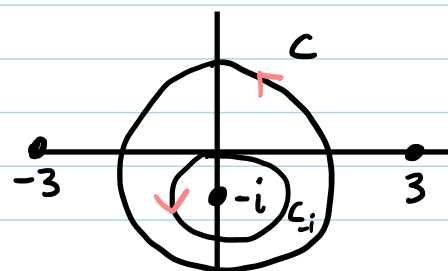
Near $z=1$:

$$\begin{aligned} \int_{C_1} \frac{3z-2}{z(z-1)} dz &= \int_{C_1} \frac{g(z)}{z-1} dz & g(z) &= \frac{3z-2}{z} \\ &= 2\pi i & g(1) &= 1 \end{aligned}$$

$$\int_C = 4\pi i + 2\pi i = 6\pi i. \quad \text{Add up contributions}$$

ex) $\int_C \frac{z}{(9-z^2)(z+i)} dz, \quad C: \text{circle } |z|=2 \quad \text{ccw}$

Replace C with $C_{-i}: |z+i|=\varepsilon \quad \text{ccw}$



$$\int_{C_{-i}} \frac{f(z)}{z+i} dz, \quad f = \frac{z}{9-z^2}, \quad f(-i) = \frac{-i}{10}$$

$$= 2\pi i \left(\frac{-i}{10} \right) \Rightarrow \int_C = \frac{\pi}{5}$$

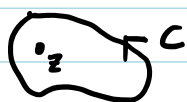
Cauchy's Derivative Formula

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} dz, \quad C: \text{any curve containing the point } z_0.$$

f : analytic inside and on C .

Proof of derivative formula:

$$\text{Cauchy Integral Formula: } f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$



$$\frac{\partial}{\partial z} f(z) = \frac{\partial}{\partial z} \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \int_C f(\zeta) \frac{\partial}{\partial z} \left(\frac{1}{(\zeta - z)} \right) d\zeta$$

$$\therefore \frac{\partial f}{\partial z} = \int_C f(\zeta) \frac{1}{(\zeta - z)^2} d\zeta \quad \checkmark$$

Higher derivatives:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$$\text{ex) Compute } \int_C \frac{\sin z}{z^2} dz, \quad C: |z|=1 \text{ CCW.}$$

$$\begin{aligned} \int_C \frac{\sin z}{z^2} &= \int_C \frac{f(z)}{z^2}, & f &= \sin z \\ & & f' &= \cos z \\ & & f'(0) &= 1 \\ &= 2\pi i f'(0) \\ &= 2\pi i. \end{aligned}$$

ex) Compute $\int_C \frac{2z^2 + z + 5}{(z+i)^3} dz$, $C: |z|=2$ ccw.

Let $f(z) = 2z^2 + z + 5$.

$$f' = 4z + 1$$

$$f'' = 4$$

$$\int_C = \int_C \frac{f(z)}{(z+i)^3} dz$$

$$= \frac{2\pi i}{2!} f''(-i)$$

$$\Rightarrow \int_C = \frac{(2\pi i)(4)}{2} = 4\pi i$$