

Liouville Theorem and Maximum Principle

Thm: (Liouville) Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be analytic and bounded. Then f is a constant function.

Proof: Suppose $|f(z)| \leq M$.

$$f'(z_0) = \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^2} dz, \quad C_R: |z-z_0| = R$$

$$|f'(z_0)| \leq \frac{1}{2\pi} \int_{C_R} \frac{|f(z)|}{|z-z_0|^2} dz$$

$$\leq \frac{M}{2\pi R^2} \text{Length}(C_R) = \frac{M}{R} \xrightarrow{R \rightarrow \infty} 0.$$

$\therefore f'(z_0) = 0$ at all points $z_0 \in \mathbb{C}$.

$\therefore f$ is constant. $\square$

Thm: Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be analytic and satisfies

$$|f(z)| \leq A + B|z|^n, \quad \text{where } A, B > 0, n \text{ positive integer.}$$

Then: $f(z)$ is a polynomial of degree n .

Proof:

$$f^{(n+1)}(z_0) = \frac{(n+1)!}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^{n+2}} dz \quad C_R: |z-z_0| = R$$

$$\begin{aligned} \text{Note: } |f(z)| &\leq A + B|z-z_0+z_0|^n \\ &\leq A + B(|z-z_0| + |z_0|)^n \end{aligned}$$

$$|f^{(n+1)}(z_0)| \leq \frac{(n+1)!}{2\pi} \max_{C_R} \left(\frac{A + B(|z-z_0| + |z_0|)^n}{|z-z_0|^{n+2}} \right) \text{Length}(C_R)$$

$$= \frac{(n+1)!}{2\pi} \left(\frac{A + B(R + |z_0|)^n}{R^{n+2}} \right) (2\pi R) \xrightarrow{R \rightarrow \infty} 0$$

Cauchy
derivative
formula

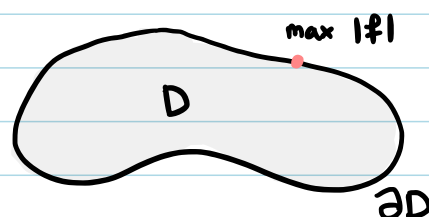
$$\therefore f^{(n+1)} \equiv 0 \text{ on } \mathbb{C}.$$

Take $n+1$ anti-derivatives $\Rightarrow f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$. $\square$

Maximum Principle

Let $f(z)$ be analytic in a bounded domain $D \subseteq \mathbb{C}$. Then:

$$|f| \leq \max_{\partial D} |f|.$$



Proof:

$$1. \nabla^2 = 4 \partial_z \partial_{\bar{z}}. \quad \text{Laplacian: } \nabla^2 = \partial_x \partial_x + \partial_y \partial_y$$

$$\begin{aligned} \partial_z \partial_{\bar{z}} &= \frac{1}{2} (\partial_x - i \partial_y) \frac{1}{2} (\partial_x + i \partial_y) \\ &= \frac{1}{4} (\partial_x \partial_x + i \partial_x \partial_y - i \partial_y \partial_x + \partial_y \partial_y) \\ &= \frac{1}{4} \nabla^2 \end{aligned}$$

$$2. \nabla^2 |f|^2 = 4 |\partial_z f|^2$$

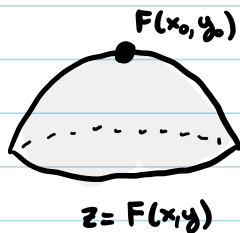
$$\begin{aligned} \nabla^2 |f|^2 &= 4 \partial_z \partial_{\bar{z}} (f \bar{f}) \\ &= 4 \partial_z (f \partial_{\bar{z}} \bar{f}) \quad \partial_{\bar{z}} f = 0 \\ &= 4 \partial_z (f \overline{\partial_z f}) \\ &= 4 \partial_z f \overline{\partial_z f} + 4 f \underbrace{\overline{\partial_z \partial_z f}}_{=0} \\ &= 4 |\partial_z f|^2. \end{aligned}$$

$$\begin{aligned} 3. \nabla^2 (|f|^2 + \varepsilon |z|^2) &= 4 |\partial_z f|^2 + 4 \varepsilon \partial_z \partial_{\bar{z}} (z \bar{z}) \\ &= 4 |\partial_z f|^2 + 4 \varepsilon. \end{aligned}$$

4. Recall second derivative test.

Suppose $F(x,y): D \rightarrow \mathbb{R}$ has a local max at (x_0, y_0) . Then:

$$(\det D^2 F)(x_0, y_0) \geq 0 \quad \text{and} \quad \begin{matrix} F_{xx}(x_0, y_0) \leq 0 \\ F_{yy}(x_0, y_0) \leq 0 \end{matrix} \quad \text{and} \quad \nabla F|_{(x_0, y_0)} = \vec{0}.$$



5. Suppose $|f|^2 + \varepsilon |z|^2$ has an interior max at $z_0 \in D$.

$$\text{By 2}^{\text{nd}} \text{ deriv test: } \nabla^2(|f|^2 + \varepsilon |z|^2)|_{z_0} \leq 0$$

$$\text{By step 3: } \nabla^2(|f|^2 + \varepsilon |z|^2)|_{z_0} > 0.$$

Cannot happen.

$$\begin{aligned} \therefore |f| + \varepsilon |z|^2 &\leq \max_{\partial D} (|f|^2 + \varepsilon |z|^2) \\ &\leq \max_{\partial D} |f|^2 + \varepsilon (\text{Diam } \Omega)^2. \end{aligned}$$

$$6. \text{ Let } \varepsilon \rightarrow 0. \quad |f|^2 \leq \max_{\partial D} |f|^2$$

$$\Rightarrow |f| \leq \max_{\partial D} |f|. \quad \square$$

Note: The "strong maximum principle" states:

Let f be analytic in $D \subseteq \mathbb{C}$. If $|f|$ attains its maximum value inside D , then f is a constant function.

will not
prove

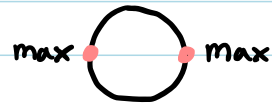
ex) Find the max of $|e^{z^2}|$ on $|z| \leq 1$.

By max princ, the max occurs on $|z|=1$.

Let $z = e^{i\varphi} \in \{|z|=1\}$.

$$\begin{aligned} |e^{z^2}| &= |e^{\cos 2\varphi + i \sin 2\varphi}| \\ &= |e^{\cos 2\varphi}| \cdot |e^{i \sin 2\varphi}| = 1 \end{aligned}$$

Max occurs when $\cos 2\varphi = 1$.
 $\varphi = 0, \pi$



$$\max_{|z| \leq 1} |e^{z^2}| = e.$$

ex) Compute $\max_{|z| \leq 1} |2z^2 + z + 1|$.

By max princ, the max occurs on $|z|=1$.

$$\text{Easier way: } |2z^2 + z + 1| \leq 2|z|^2 + |z| + 1$$

$$\therefore \max_{|z| \leq 1} |2z^2 + z + 1| \leq 2 + 1 + 1 = 4.$$

But by inspection, $|2z^2 + z + 1| = 4$ at $z = 1$.

$\therefore$ max attained at $z = 1$.

$\therefore$ max = 4.

ex) Compute $\max_{|z| \leq 1} \left| \frac{z^2 + 1}{z - 2} \right|$.

Only need to check $|z| = 1$. Let $z = e^{it}$.

$$\max_t \left| \frac{e^{2it} + 1}{e^{it} - 2} \right| = ? \quad \text{Let's maximize the square.}$$

$$|e^{2it} + 1|^2 = |\cos(2t) + i \sin(2t) + 1|^2$$

$$= (\cos(2t) + 1)^2 + \sin^2(2t)$$

$$= 2 \cos 2t + 2 = 4 \cos^2 t$$

$$\cos^2 t = \frac{1}{2}(\cos t + 1)$$

$$|e^{it} - 2|^2 = |\cos t + i \sin t - 2|^2$$

$$= (\cos t - 2)^2 + \sin^2 t$$

$$= -4 \cos t + 5$$

$$\Rightarrow \left| \frac{e^{2it} + 1}{e^{it} - 2} \right|^2 = \frac{4 \cos^2 t}{5 - 4 \cos t} = \frac{\bigcirc}{\bigcirc} \quad \begin{array}{l} \leftarrow \text{biggest value is 4} \\ \leftarrow \text{smallest value is } 5 - 4 = 1 \end{array}$$

$$\leq 4.$$

$$\text{This is attained at } t=0: \frac{4 \cos^2 0}{5 - 4 \cos 0} = 4.$$

$$\therefore \max_t \left| \frac{e^{2it} + 1}{e^{it} - 2} \right|^2 = 4.$$

$$\Rightarrow \max_{|z| \leq 1} \left| \frac{z^2 + 1}{z - 2} \right| = 2.$$