

Taylor Series

If f is analytic at z_0 ,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$$

$$= f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!} (z-z_0)^2 + \dots$$

is the Taylor series at z_0 .

ex) Taylor series of e^z at 0:

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \dots \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \end{aligned}$$

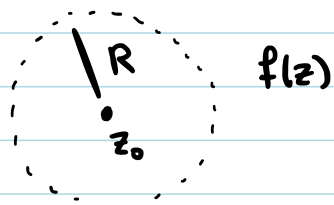
ex) Taylor series of $\frac{1}{1-z}$ at 0.

$$\begin{aligned} \frac{1}{1-z} &= 1 + z + z^2 + z^3 + \dots \\ &= \sum_{n=0}^{\infty} z^n \end{aligned}$$

Obviously nonsense if we plugin $z=2$:

$$1 + 2 + 2^2 + 2^3 + 2^4 + \dots = \infty, \text{ not } \frac{1}{1-2}.$$

Thm: If f is analytic on $D = |z - z_0| < R$, then its Taylor series converges on D .



Proof: Let z with $|z - z_0| < R$. Squeeze in: $|z - z_0| < s < r < R$.

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta, \quad C = \text{circle } \{\zeta : |\zeta - z_0| = r\}$$

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)}$$

$$= \frac{1}{(\zeta - z_0) \left(1 - \frac{z - z_0}{\zeta - z_0}\right)}$$

$$= \frac{1}{(\zeta - z_0)} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n$$

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$$

valid on $|w| < 1$

Convergent on $\left| \frac{z - z_0}{\zeta - z_0} \right| < 1$.

$$\text{On } C: \frac{|z - z_0|}{|\zeta - z_0|} < \frac{s}{r} < 1 \quad \checkmark$$

$$\therefore f(z) = \frac{1}{2\pi i} \left(\int_C \frac{f(\zeta)}{\zeta - z} d\zeta \right) \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n!} f^{(n)}(z_0) \right) (z - z_0)^n.$$

Cauchy derivative formula.

Formula valid for any z with $|z - z_0| < R$. $\square$

Cor: Radius of convergence of the Taylor series of $f(z)$ at z_0 is the distance to nearest singularity.

$$\text{ex)} \quad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{converges } \forall z$$

$$\text{ex)} \quad \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

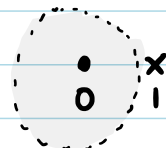
$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad \text{converges } \forall z$$

$$\text{ex)} \quad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \quad \text{converges } \forall z$$

$$\text{ex)} \quad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{converges on largest disk } |z| < R \quad \text{where } f(z) = \frac{1}{1-z} \text{ is analytic.}$$

$\Rightarrow$ converges on $|z| < 1$.



$$\text{ex)} \quad \frac{1}{1+z} = 1 - z + z^2 - z^3 + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n z^n \quad \text{converges on } |z| < 1$$

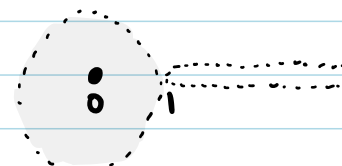
$$\text{ex)} \quad (1+z)^{-n} = \sum_{k=0}^{\infty} (-1)^k \frac{n(n+1)\dots(n+k-1)}{k!} z^k \quad \text{on } |z| < 1$$

$$= 1 - nz + \dots$$

$$\text{ex)} \quad \text{Log}(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \dots$$

$$= -\sum_{n=1}^{\infty} \frac{z^n}{n} \quad \text{on } |z| < 1.$$

$$\text{Re}(1-z) \leq 0 \Rightarrow \text{Re } z \geq 1.$$



ex) Find Taylor series of $\frac{1+z}{1-z}$ at $z_0 = i$, and give radius of convergence.

Radius = $|1-i|$ distance from i to singularity.
 $= \sqrt{2}$.



On $|z-i| < \sqrt{2}$, the series is:

$$\begin{aligned}\frac{1+z}{1-z} &= \frac{1+i+(z-i)}{1-i-(z-i)} = \frac{(1+i)+(z-i)}{1-i} \left(1 - \frac{z-i}{1-i}\right)^{-1}, \quad c = \frac{z-i}{1-i} \\ & \quad |c| < 1 \\ &= \left(\frac{1+i}{1-i} + c \right) \left(1 + c + c^2 + c^3 + \dots \right) \\ & \quad = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{2i}{2} = i\end{aligned}$$

$$= (i+c)(1+c+c^2+\dots)$$

$$= i + i(c+c^2+\dots) + (c+c^2+\dots)$$

$$= i + (1+i)(c+c^2+\dots)$$

$$= i + \sum_{n=1}^{\infty} \frac{(1+i)}{(1-i)^n} (z-i)^n. \text{ valid on } |z-i| < \sqrt{2}$$

Def: A power series at z_0 is a series of the form:

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad a_n \in \mathbb{C}.$$

Ratio Test: Consider the series $\sum_{n=0}^{\infty} c_n$:

- If $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = L < 1$, the series converges.
- If $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = L > 1$, the series diverges.

ex) Find radius of convergence of $\sum_{n=0}^{\infty} \frac{i^n n^2}{2^{2n}} z^n$.

$$\text{Let } c_n = \frac{i^n n^2 z^n}{2^{2n}}.$$

$$\left| \frac{c_{n+1}}{c_n} \right| = \frac{(n+1)^2 |z|^{n+1}}{2^{2(n+1)}} \frac{2^{2n}}{n^2 |z|^n} \xrightarrow{n \rightarrow \infty} \frac{|z|}{4}.$$

$\therefore$ Series converges on $|z| < 4$.

Recall: Termwise differentiation and integration is valid inside radius of convergence.

$$\text{ex) } \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{on } |z| < 1$$

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1} \quad \text{on } |z| < 1.$$

Remark: If $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges on $|z-z_0| < R$, then

$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n$ is an analytic function on $|z-z_0| < R$.

$$\frac{\partial}{\partial \bar{z}} f = \frac{\partial}{\partial \bar{z}} \sum a_n (z-z_0)^n = \sum a_n \frac{\partial}{\partial \bar{z}} (z-z_0)^n = 0. \quad \checkmark$$

Back to Taylor Series:

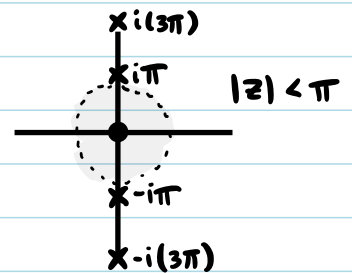
ex) Find first 3 terms of Taylor series of $\frac{1}{1+e^z}$ at $z_0 = 0$.

What is the radius of convergence?

Singularities at $1+e^z = 0 \Rightarrow z = \pm i\pi, \pm 3i\pi, \dots$

Radius of convergence = distance between 0 and nearest singularity.
 $= \pi$.

Taylor series converges on $|z| < \pi$.



$$f' = -(1+e^z)^{-2} e^z$$

$$f'' = 2(1+e^z)^{-3} e^z e^z - (1+e^z)^{-2} e^z$$

$$f(0) = 1/2$$

$$f'(0) = -1/4$$

$$f''(0) = 2 \cdot 2^{-3} - 2^{-2} = 0$$

$$f''' = -6(1+e^z)^{-4} e^z e^z e^z + 2(1+e^z)^{-3} 2e^{2z} + 2(1+e^z)^{-3} e^z e^z - (1+e^z)^{-2} e^z$$

$$f'''(0) = -3 \cdot 2^{-3} + 2^{-1} + 2^{-2} - 2^{-2}$$

$$= -\frac{3}{8} + \frac{4}{8} = \frac{1}{8}$$

$$\Rightarrow (1+e^z)^{-1} = f(0) + f'(0)z + \frac{f''(0)}{2} z^2 + \frac{f'''(0)}{6} z^3 + \dots$$

$$= \frac{1}{2} - \frac{z}{4} + \frac{1}{48} z^3 + \dots$$

ex) Find radius of convergence of Taylor series of $\sec z = \frac{1}{\cos z}$ at $z = 0$.

Singularities are when $\cos z = 0 \Rightarrow z = \pm \pi/2, \pm 3\pi/2, \dots$

Taylor series converges on $|z| < \pi/2$.