

Laurent Series

Let $f(z)$ be an analytic function with a singularity at 0.
How to write a series expansion at 0?

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \dots \quad \text{not helpful if } f(0) = \infty.$$

Can still expand $f(z)$ near a singularity, and the expansion is called a Laurent series.

ex) Laurent series of $f(z) = \frac{z - \sin z}{z^5}$ at $z_0 = 0$.

$$f(z) = \frac{1}{z^5} \left(z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \right) \right) \quad \text{expand } \sin z$$

$$= \frac{z^{-2}}{3!} - \frac{1}{5!} + \frac{z^2}{7!} - \dots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-4} \quad \text{convergent on } 0 < |z| < \infty.$$

ex) Laurent series of $e^{1/z}$ at $z_0 = 0$.

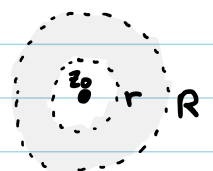
$$e^{1/z} = 1 + z^{-1} + \frac{1}{2!} z^{-2} + \frac{1}{3!} z^{-3} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}.$$

Region of convergence: $0 < |z| < \infty$.

Theorem: Let $f(z)$ be analytic over $r < |z - z_0| < R$.

There exists a unique series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

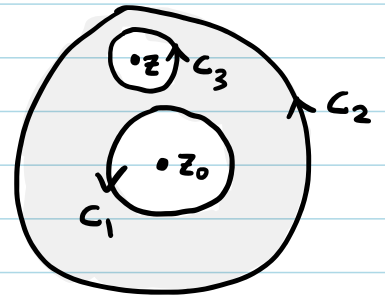


Converging on $r < |z - z_0| < R$.

Proof of Laurent Series: Suppose $f(z)$ is analytic on $r < |z - z_0| < R$.
Introduce $r < \rho_{in} < |z - z_0| < \rho_{out} < R$.

Green's Thm:

$$\begin{aligned} 0 &= 2i \iint \frac{\partial}{\partial \bar{\tau}} \left(\frac{f(\tau)}{\tau - z} \right) dA \\ &= \int_{C_2} \frac{f(\tau)}{\tau - z} d\tau - \int_{C_1} \frac{f(\tau)}{\tau - z} d\tau \\ &\quad - \underbrace{\int_{C_3} \frac{f(\tau)}{\tau - z} d\tau}_{= 2\pi i f(z)} \end{aligned}$$



$$\begin{aligned} C_1 &= \{ |\tau - z_0| = \rho_{in} \} \\ C_2 &= \{ |\tau - z_0| = \rho_{out} \} \\ C_3 &= \{ |\tau - z_0| = \epsilon \} \end{aligned}$$

$$\therefore f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\tau)}{\tau - z} d\tau - \frac{1}{2\pi i} \int_{C_1} \frac{f(\tau)}{\tau - z} d\tau.$$

(I) (II)

Over C_2 :

$$\begin{aligned} \frac{1}{\tau - z} &= \frac{1}{\tau - z_0} \left(1 - \left(\frac{z - z_0}{\tau - z_0} \right) \right)^{-1}, \quad \left| \frac{z - z_0}{\tau - z_0} \right| = \frac{|z - z_0|}{\rho_{out}} < 1. \\ &\stackrel{(I)}{=} \frac{1}{\tau - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\tau - z_0} \right)^n. \quad \leftarrow \text{series valid} \end{aligned}$$

Over C_1 :

$$\begin{aligned} \frac{1}{\tau - z} &= \frac{-1}{z - z_0} \left(1 - \left(\frac{\tau - z_0}{z - z_0} \right) \right)^{-1}, \quad \left| \frac{\tau - z_0}{z - z_0} \right| = \frac{\rho_{in}}{|z - z_0|} < 1. \\ &\stackrel{(II)}{=} \frac{-1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{\tau - z_0}{z - z_0} \right)^n \quad \leftarrow \text{series valid} \end{aligned}$$

Sub (I), (II)

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_2} \frac{f(\tau)}{\tau - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\tau - z_0} \right)^n d\tau \\ &\quad + \frac{1}{2\pi i} \int_{C_1} \frac{f(\tau)}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{\tau - z_0}{z - z_0} \right)^n d\tau \end{aligned}$$

$$\therefore f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=0}^{\infty} b_n (z-z_0)^{-n-1},$$

$$a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}} d\zeta$$

$$b_n = \frac{1}{2\pi i} \int_{C_1} f(\zeta) (\zeta-z_0)^n d\zeta. \quad \square$$

ex) Find the Laurent series for $f(z) = \frac{1}{z(z-1)(z-2)}$ at $z_0 = 0$

over: a) $0 < |z| < 1$
b) $1 < |z| < 2$.

use partial fractions:

$$\frac{1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

$$\Rightarrow 1 = A(z-1)(z-2) + Bz(z-2) + Cz(z-1)$$

$$\Rightarrow B = -1 \quad (z=1)$$

$$A = 1/2 \quad (z=0)$$

$$C = 1/2 \quad (z=2)$$

$$\Rightarrow f(z) = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$$

$$(a) f(z) = \frac{1}{2z} + \frac{1}{1-z} - \frac{1}{4} \frac{1}{1-z/2}$$

$$= \frac{1}{2z} + \sum_{n=0}^{\infty} z^n - \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \quad \text{valid since } |z| < 1.$$

$$(b) f(z) = \frac{1}{2z} - \frac{1}{z} \frac{1}{1-1/2} - \frac{1}{4} \frac{1}{1-z/2}$$

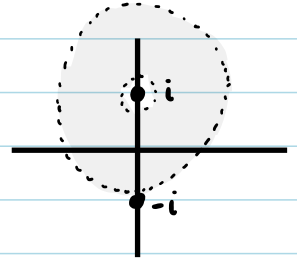
$$= \frac{1}{2z} - \frac{1}{z} \sum_{n=0}^{\infty} z^{-n} - \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \quad \text{valid since } |z|^{-1} < 1, \frac{|z|}{2} < 1.$$

ex) Find the Laurent series of $f(z) = \frac{z}{z^2+1}$ at $z_0 = i$.

Region of convergence?

Singularities are at $\pm i$. Distance from $+i$ to $-i$ = 2.

Region of convergence: $0 < |z-i| < 2$.



$$\frac{z}{(z+i)(z-i)} = \frac{A}{z+i} + \frac{B}{z-i}$$

$$z = A(z-i) + B(z+i)$$

$$\Rightarrow A = \frac{1}{2} \quad (z=-i)$$

$$B = \frac{1}{2} \quad (z=+i)$$

$$f(z) = \frac{1}{2(z+i)} + \frac{1}{2(z-i)}$$

$$= \frac{1}{2(z-i+i+i)} + \frac{1}{2(z-i)}$$

$$= \frac{1}{2(2i)\left(1 + \underbrace{\left(\frac{z-i}{2i}\right)}_{| \cdot | < 1}\right)} + \frac{1}{2(z-i)}$$

$$= \frac{1}{4i} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-i}{2i}\right)^n + \frac{1}{2(z-i)}$$

Zeros and poles

Def: Let $f(z)$ be analytic near z_0 with $f(z_0) = 0$.

We say z_0 is a zero of order m if its Taylor series is

$$f(z) = \underbrace{a_m}_{\neq 0} (z-z_0)^m + a_{m+1} (z-z_0)^{m+1} + a_{m+2} (z-z_0)^{m+2} + \dots$$

Def: Let $f(z)$ be analytic on $0 < |z-z_0| < R$.

We say z_0 is a pole of order $m > 0$ if its Laurent series is

$$f(z) = \underbrace{a_{-m}}_{\neq 0} (z-z_0)^{-m} + a_{-m+1} (z-z_0)^{-m+1} + a_{-m+2} (z-z_0)^{-m+2} + \dots$$

If $m=0$, say z_0 is a removable singularity.

$$\text{If } f(z) = \sum_{m=0}^{\infty} a_{-m} (z-z_0)^{-m} + \sum_{m=1}^{\infty} a_m (z-z_0)^m$$

with infinitely many non-zero a_{-m} , say z_0 is an essential singularity.

ex) $f(z) = z^3 + iz^5 + 3z^6$
zero of order 3.

ex) $f(z) = \frac{3}{z^8} + \frac{6}{z^3} + \frac{2}{z} + 1 + z^4$
pole of order 8.

ex) $f(z) = e^{1/z} = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{1}{z^n}$ essential singularity.

ex) Classify the singularity of $f(z) = \frac{1 - \cos z^2}{z^6}$.

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

$$f(z) = z^{-6} \left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \dots \right) = -\frac{z^{-4}}{2!} + \frac{z^{-2}}{4!} - \dots \quad \text{pole of order 4.}$$

ex) Classify the singularity of $f(z) = \frac{1}{z \tan z}$. Find first 2 terms of Laurent expansion at 0.

$$\tan z = \frac{\sin z}{\cos z}$$

$$(1-u)^{-1} = 1+u+\dots$$

$$\begin{aligned} &= \frac{z - \frac{z^3}{3!} + \dots}{1 - \frac{z^2}{2!} + \dots} = \left(z - \frac{z^3}{3!} + \dots \right) \left(1 + \frac{z^2}{2} + \dots \right) \\ &= z + \left(\frac{1}{2} - \frac{1}{3!} \right) z^3 + \dots \\ &\quad \quad \quad = \frac{1}{3} \end{aligned}$$

$$z \tan z = z^2 + \frac{z^4}{3} + \dots$$

$$(z \tan z)^{-1} = \left(z^2 \left(1 + \frac{z^2}{3} + \dots \right) \right)^{-1}$$

$$= z^{-2} \left(1 - \frac{z^2}{3} + \dots \right)$$

$$= z^{-2} - \frac{1}{3} + \dots \quad \text{pole of order 2.}$$

Remark: Suppose $f(z) = \frac{P(z)}{Q(z)}$ with: P, Q analytic near z_0
 $: P(z_0) \neq 0$
 $: Q(z_0) = 0$
 $: Q'(z_0) \neq 0$

Then f has a pole of order 1 (i.e. simple pole) at z_0 .

$$a_{-1} = \frac{P(z_0)}{Q'(z_0)}.$$

$$\text{If } z_0 = 0, f = \frac{a_0 + a_1 z + a_2 z^2 + \dots}{0 + b_1 z + b_2 z^2 + \dots} = \frac{(a_0 + a_1 z + \dots)}{b_1 z} \left(1 + \frac{b_2}{b_1} z + \dots \right)^{-1}$$

$$= \left(\frac{a_0}{b_1} z^{-1} + \frac{a_1}{b_1} + \dots \right) \left(1 - \frac{b_2}{b_1} z + \dots \right) = \frac{a_0}{b_1} z^{-1} + \dots$$

Remark: Suppose

$$f(z) = \frac{P(z)}{Q(z)} \quad \text{with: } P, Q \text{ analytic near } z_0$$
$$\begin{aligned} &: P(z_0) \neq 0 \\ &: Q(z_0) = 0, Q'(z_0) = 0, \dots, Q^{(m-1)}(z_0) = 0 \\ &: Q^{(m)}(z_0) \neq 0 \end{aligned}$$

Then f has a pole of order m at z_0 . Same proof as above

ex) Compute the order of the pole at $z = \frac{\pi}{2}$ for $f(z) = \frac{1}{\cos z}$.

$$f(z) = \frac{1}{Q(z)}, \quad Q(z) = \cos z$$
$$Q\left(\frac{\pi}{2}\right) = 0$$
$$Q'\left(\frac{\pi}{2}\right) = -1$$

$\Rightarrow f$ has a simple pole at $z = \frac{\pi}{2}$.

ex) Classify the singularities of $f(z) = \frac{z^2(z-1)}{(\sin \pi z)^2}$.

• $z = 0$.

$$\sin \pi z = \pi z - \frac{(\pi z)^3}{3!} + \dots = \pi z \left(1 - \frac{(\pi z)^2}{3!} + \dots\right)$$

$$f(z) = \frac{z^2(z-1)}{(\pi z)^2} \left(1 + \frac{(\pi z)^2}{3!} + \dots\right)^2 = \frac{1}{\pi^2} (z-1) + \dots$$

removable singularity. $f(0) = -1/\pi^2$.

• $z = 1$.

Expand $Q = (\sin \pi z)^2$ at $z_0 = 1$.

$$Q(1) = 0$$

$$Q'(z) = 2\pi \sin \pi z \cos \pi z \Rightarrow Q'(1) = 0$$

$$Q''(z) = 2\pi^2 \cos^2 \pi z - 2\pi^2 \sin^2 \pi z \Rightarrow Q''(1) = 2\pi^2$$

$$\Rightarrow (\sin \pi z)^2 = \frac{Q''(1)}{2} (z-1)^2 + \dots = \pi^2 (z-1)^2 + \dots$$

Expand $P = z^2(z-1)$ at $z_0 = 1$.

$$P(1) = 0$$

$$P'(z) = 2z(z-1) + z^2 \Rightarrow P'(1) = 1.$$

$$\Rightarrow z^2(z-1) = P'(1)(z-1) + \dots \\ = (z-1) + \dots$$

$$\Rightarrow f(z) = \frac{(z-1) + \dots}{\pi^2(z-1)^2 + \dots} = \frac{1}{\pi^2} (z-1)^{-1} + \dots \text{ near } z=1. \\ \Rightarrow \text{pole of order 1.}$$

$$\bullet z_0 = -1, \pm 2, \pm 3, \pm 4, \dots$$

$$f(z) = \frac{P(z)}{Q(z)}$$

$$P(z_0) \neq 0, Q(z_0) = 0,$$

$$Q'(z_0) = 0$$

$$Q''(z_0) = 2\pi^2$$

calculated above

$\Rightarrow f$ has a pole of order 2 at z_0 .