

Mathematics 305 — Final Exam

Practice Final A

- The test consists of 12 pages and 11 questions worth a total of 60 marks.
- Electronic devices of any kind (including calculators, cell phones, etc.) are forbidden.
- No work on this page will be marked.
- If you require extra paper then put up your hand and ask your instructor. You must indicate your name, student number, test number and question number on any extra pages.

Student number								
Section								
Name								
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1. (a) 4 marks Find all solutions to $e^{z^2} = -1$.

Solution: The solution to $e^{z^2} = e^{i\pi}$ is

$$z^2 = i\pi + i(2\pi k), \quad k = 0, \pm 1, \pm 2, \dots$$

Suppose first that $k \geq 0$. In this case, then

$$z^2 = \pi(2k + 1)e^{i\pi/2}, \quad k = 0, 1, 2, \dots$$

and so

$$z = \pm \sqrt{\pi(2k + 1)}e^{i\pi/4}.$$

If instead $k \leq -1$, then if we let $k = -m - 1$ we have

$$z^2 = \pi(2m + 1)e^{-i\pi/2}, \quad m = 1, 2, 3, \dots$$

and so

$$z = \pm \sqrt{\pi(2m + 1)}e^{-i\pi/4}, \quad m = 0, 1, 2, \dots$$

Combining both cases gives the final answer:

$$z = \pm \sqrt{\pi(2k + 1)}e^{\pm i\pi/4}, \quad k = 0, 1, 2, \dots$$

- (b) 4 marks Find all solutions to $\sqrt{z} = 1 - i$, where $\sqrt{z}$ denotes the principal branch.

Solution: We write the right-hand side in polar form.

$$1 - i = \sqrt{2}e^{-i\pi/4}.$$

If $z = |z|e^{i\varphi}$ for $-\pi < \varphi \leq \pi$, then $\sqrt{z} = \sqrt{|z|}e^{i\varphi/2}$. Matching this with $1 - i$ gives

$$z = 2e^{-i\pi/2}$$

which is just $z = -2i$.

2. 6 marks Consider the branch

$$\sqrt{z^2 - 1} = r_1^{1/2} r_2^{1/2} e^{i(\theta_1 + \theta_2)/2}, \quad 0 \leq \theta_1 < 2\pi, \quad -\pi < \theta_2 \leq \pi,$$

with $z - 1 = r_1 e^{i\theta_1}$ and $z + 1 = r_2 e^{i\theta_2}$, and define the inverse cosine function by

$$\cos^{-1}(z) = -i \operatorname{Log} \left(z + \sqrt{z^2 - 1} \right),$$

where Log denotes the principal branch of the logarithm. Using this definition, compute $\cos^{-1}(0)$ and find the image of

$$S = \{-1 \leq \operatorname{Re}(z) \leq 1, \quad \operatorname{Im}(z) = 0\}.$$

In other words, find $S' = \cos^{-1}(S)$.

Solution: If $z = x$ with $-1 \leq x \leq 1$, then

$$z - 1 = (1 - x)e^{i\pi}, \quad z + 1 = (1 + x)e^{i0}.$$

Therefore

$$\sqrt{z^2 - 1} = (1 - x)^{1/2} (1 + x)^{1/2} e^{i\pi/2} = i\sqrt{1 - x^2}.$$

Next, we compute

$$\operatorname{Log}(z + \sqrt{z^2 - 1}) = \operatorname{Log}(x + i\sqrt{1 - x^2}).$$

In polar form we have

$$x + i\sqrt{1 - x^2} = |x + i\sqrt{1 - x^2}| e^{i\varphi} = e^{i\varphi},$$

where

$$\cos \varphi = x, \quad \sin \varphi = \sqrt{1 - x^2}.$$

Since $x + i\sqrt{1 - x^2}$ traces out the upper unit semicircle as x goes from -1 to 1, we see that φ takes all values in $[0, \pi]$. Thus

$$\cos^{-1}(x) = -i \operatorname{Log} \left(x + \sqrt{x^2 - 1} \right) = -i(i\varphi) = \varphi,$$

and

$$\cos^{-1}(S) = [0, \pi].$$

When $x = 0$, then $i = e^{i\varphi}$ implies $\varphi = \pi/2$ and so

$$\cos^{-1}(0) = \frac{\pi}{2}.$$

3. 5 marks Let $f(z)$ be an analytic function. Suppose

$$|f(e^{i\theta})| \leq 3$$

for all angles $0 \leq \theta \leq 2\pi$. Further suppose $f(z)$ has a zero of order 1 at $z = 0$. Find the largest possible value of $|f(1/2)|$. Hint: consider $g(z) = f(z)/z$.

Solution: The function $g(z) = f(z)/z$ is analytic, and

$$|g(e^{i\theta})| = |e^{-i\theta}| |f(e^{i\theta})| \leq 3.$$

Therefore

$$\max_{|z|=1} |g| \leq 3$$

which implies $|g(1/2)| \leq 3$ by the maximum principle. Thus

$$\left| \frac{f(1/2)}{1/2} \right| \leq 3 \implies |f(1/2)| \leq \frac{3}{2}.$$

Therefore the largest possible value is $3/2$. This bound is attained by e.g. $f(z) = 3z$.

4. 4 marks Evaluate

$$\int_C \frac{1+z}{z^2} dz$$

where C is the upper semi-circle $C = \{|z| = 1, \operatorname{Im} z \geq 0\}$ running clockwise from $z = -1$ to $z = 1$.

Solution: Let

$$f(z) = \frac{1+z}{z^2} = z^{-2} + z^{-1}.$$

We parametrize the curve C by $z(t) = e^{it}$ with $t : \pi \rightarrow 0$. The integral is

$$\int_C f dz = \int_{\pi}^0 (e^{-2it} + e^{-it})(ie^{it} dt)$$

which becomes

$$\int_C f dz = -i \int_0^{\pi} (e^{-it} + 1) dt = -i \left[\frac{e^{-it}}{-i} + t \right]_0^{\pi}$$

and evaluates to

$$\int_C f dz = -i(ie^{-i\pi} - i + \pi) = -2 - i\pi.$$

5. 4 marks Compute $\int_C z^2 \sin \frac{1}{z} dz$ where C is the counterclockwise unit circle $|z| = 1$.

Solution: The Laurent expansion of the integrand is

$$z^2 \sin \frac{1}{z} = z^2 \left[z^{-1} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \cdots \right] = z - \frac{1}{6} \frac{1}{z} + \frac{1}{5!} \frac{1}{z^3} + \cdots$$

The line integral of z^m over C is zero unless $m = -1$, in which case the integral is $2\pi i$. Therefore the only term that contributes to the integral is $-\frac{1}{6}z^{-1}$. The result is

$$\int_C z^2 \sin \frac{1}{z} dz = -\frac{1}{6}(2\pi i) = -\frac{\pi i}{3}.$$

6. 4 marks Evaluate $\int_C \frac{1}{z^3(z+6)} dz$ where C is the counterclockwise circle $|z| = 2$.

Solution: The only pole contained in $|z| \leq 2$ is $z = 0$. By Cauchy's derivative formula,

$$\int_C \frac{1}{z^3(z+6)} dz = \frac{2\pi i}{2!} g''(0), \quad g(z) = \frac{1}{z+6}.$$

We compute two derivatives of $g(z)$.

$$g' = -(z+6)^{-2}, \quad g'' = 2(z+6)^{-3}.$$

Therefore $g''(0) = 2 \cdot 6^{-3} = 1/108$. Hence

$$\int_C \frac{1}{z^3(z+6)} dz = \frac{\pi i}{108}.$$

7. 6 marks Find the first three terms of the Laurent series centred at $z_0 = 3$ of

$$f(z) = \frac{1}{(z-1)^2(z-3)}$$

valid in $0 < |z-3| < 2$.

Solution: We rewrite f as

$$f(z) = \frac{1}{z-3} \left[\frac{1}{(2+(z-3))^2} \right] = \frac{1}{4(z-3)} \left[\frac{1}{(1+(\frac{z-3}{2}))^2} \right].$$

The formula for a geometric series is

$$\frac{1}{1+w} = 1 - w + w^2 - w^3 + w^4 - \dots$$

and its derivative is

$$-\frac{1}{(1+w)^2} = -1 + 2w - 3w^2 + 4w^3 - \dots,$$

and this holds for $|w| < 1$. Thus

$$\frac{1}{(1+(\frac{z-3}{2}))^2} = 1 - 2\left(\frac{z-3}{2}\right) + 3\left(\frac{z-3}{2}\right)^2 - 4\left(\frac{z-3}{2}\right)^3 + \dots$$

over $|z-3| < 2$. Therefore

$$f = \frac{1}{4(z-3)} \left[1 - 2\left(\frac{z-3}{2}\right) + 3\left(\frac{z-3}{2}\right)^2 - 4\left(\frac{z-3}{2}\right)^3 + \dots \right]$$

The first three terms are

$$f = \frac{1}{4}(z-3)^{-1} - \frac{1}{4} + \frac{3}{16}(z-3) + \dots$$

8. 5 marks Find the number of roots, counted with multiplicity, of

$$f(z) = 2z^5 + 5z^3 + z + 1$$

in the region $\{1 < |z| < 2\}$.

Solution: On $|z| = 2$, we have

$$|2z^5| = 2^6 = 64$$

while

$$|5z^3 + z + 1| \leq 5|z|^3 + |z| + 1 = 5 \cdot 2^3 + 2 + 1 = 43,$$

hence

$$|5z^3 + z + 1| < |2z^5| \quad \text{on} \quad |z| = 2.$$

By Rouché's theorem, $f(z) = 2z^5 + 5z^3 + z + 1$ has the same number of zeroes counted with multiplicity inside $|z| < 2$ as $2z^5$, namely 5.

Next, we consider the region $|z| < 1$. On $|z| = 1$, we have

$$|5z^3| = 5$$

while

$$|2z^5 + z + 1| \leq 2|z|^5 + |z| + 1 = 2 + 1 + 1 = 4.$$

Hence

$$|2z^5 + z + 1| < |5z^3| \quad \text{on} \quad |z| = 1.$$

By Rouché's theorem, $f(z)$ has the same number of zeroes counted with multiplicity inside $|z| < 1$ as $5z^3$, namely 3.

Putting everything together, we conclude that $f(z)$ has

$$5 - 3 = 2$$

zeros inside $1 < |z| < 2$, counted with multiplicity.

9. 6 marks Calculate

$$\int_0^\pi \frac{2}{5 + 2 \cos \theta} d\theta$$

by complex contour integration.

Solution: This integral is half the integral from 0 to 2π . We let $z = e^{i\varphi}$, so that $dz = iz d\varphi$ and $\cos \varphi = \frac{1}{2}(z + z^{-1})$. The integral becomes

$$\frac{1}{2} \int_{|z|=1} \frac{2}{5 + z + z^{-1}} \frac{dz}{iz}.$$

This simplifies to

$$\int_{|z|=1} \frac{-i}{z^2 + 5z + 1} dz.$$

Write

$$Q(z) = z^2 + 5z + 1$$

The roots of Q are

$$z = \frac{-5 \pm \sqrt{25 - 4}}{2} \implies z = -\frac{5}{2} \pm \frac{\sqrt{21}}{2}.$$

The root $z = -\frac{5}{2} - \frac{\sqrt{21}}{2}$ is evidently outside the loop $|z| = 1$. Since $16 < 21 < 25$, we have that $4 < \sqrt{21} < 5$ and so $z = -\frac{5}{2} + \frac{\sqrt{21}}{2}$ is inside $|z| = 1$. By the residue theorem,

$$\begin{aligned} \int_{|z|=1} -\frac{i}{Q} dz &= (2\pi i) \operatorname{Res}\left(\frac{-i}{Q}, -\frac{5}{2} + \frac{\sqrt{21}}{2}\right) \\ &= -(2\pi i) \frac{i}{Q'(-\frac{5}{2} + \frac{\sqrt{21}}{2})} \\ &= (2\pi) \frac{1}{\sqrt{21}} \end{aligned}$$

since $Q'(z) = 2z + 5$. Therefore

$$\int_0^\pi \frac{2}{5 + 2 \cos \theta} d\theta = \frac{2\pi}{\sqrt{21}}.$$

10. 6 marks Calculate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 2x + 2} dx$$

by complex contour integration.

Solution: We complexify the integrand by letting

$$f(z) = \frac{e^{iz}}{z^2 + 2z + 2},$$

where $\sin z$ has been exchanged for e^{iz} for future application of Jordan's lemma. The singularities are located at

$$z = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 2}}{2}$$

which simplifies to $z = -1 \pm i$. We close the loop by introducing an upper semi circle C_R which runs along $|z| = R$ from $+R$ to $-R$. By the residue theorem

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + 2x + 2} dx + \lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{iz}}{z^2 + 2z + 2} dz = (2\pi i) \text{Res}(f, -1 + i).$$

The residue is

$$\begin{aligned} (2\pi i) \text{Res}(f, -1 + i) &= (2\pi i) \frac{e^{i(-1+i)}}{2(-1+i) + 2} \\ &= \pi e^{-1-i}. \end{aligned}$$

By Jordan's lemma, we can estimate

$$\left| \int_{C_R} \frac{e^{iz}}{z^2 + 2z + 2} dz \right| \leq \frac{1}{R^2 - 2R - 2} \cdot \pi \rightarrow 0$$

as $R \rightarrow \infty$. Therefore the loop integral becomes

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + 2x + 2} dx = \pi e^{-1-i}.$$

We take the imaginary part to retrieve $\sin x$. The right-hand side is

$$\pi e^{-1-i} = \pi e^{-1} (\cos 1 - i \sin 1)$$

and hence

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 2x + 2} dx = -\frac{\pi \sin 1}{e}.$$

11. 6 marks Calculate

$$\int_0^\infty \frac{\sqrt{x} \ln x}{x^2 + 4} dx$$

by complex contour integration.

Solution: We complexify the integrand and consider the analytic function

$$f(z) = \frac{P}{Q} = \frac{\sqrt{z} \operatorname{Log}(z)}{z^2 + 4}.$$

Here $\sqrt{z}$ and $\operatorname{Log}(z)$ denote the principal branches. The poles are at $z = \pm 2i$, and this function is undefined at $z = 0$. Consider the closed loop

$$C = C_R + C_{\text{left}} + C_\epsilon + C_{\text{right}}$$

where C_{left} runs along the x-axis from $-R$ to $-\epsilon$, C_ϵ runs from $-\epsilon$ to $+\epsilon$ along $|z| = \epsilon$, C_{right} runs from $+\epsilon$ to $+R$, and C_R runs from $+R$ along $|z| = R$ until it reaches $-R$. This contour C encloses only the pole $2i$.

By the residue theorem

$$\begin{aligned} \int_C \frac{\sqrt{z} \operatorname{Log}(z)}{z^2 + 4} dz &= (2\pi i) \operatorname{Res}(f, 2i) \\ &= (2\pi i) \frac{P(2i)}{Q'(2i)} \\ &= (2\pi i) \frac{\sqrt{2}e^{i\pi/4}(\ln 2 + i\pi/2)}{4i} \\ &= \frac{\sqrt{2}\pi}{2} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) (\ln 2 + i \frac{\pi}{2}) \\ &= \frac{\pi}{2} \left(\ln 2 - \frac{\pi}{2} + i \left(\frac{\pi}{2} + \ln 2 \right) \right). \end{aligned}$$

The integral over C_R goes to zero:

$$\left| \int_{C_R} \frac{\sqrt{z} \operatorname{Log}(z)}{z^2 + 4} dz \right| \leq \frac{R^{1/2}(\ln R + \pi)}{R^2 - 4} \cdot \pi R \rightarrow 0$$

as $R \rightarrow \infty$. The integral over C_ϵ also goes to zero:

$$\left| \int_{C_\epsilon} \frac{\sqrt{z} \operatorname{Log}(z)}{z^2 + 4} dz \right| \leq \frac{\epsilon^{1/2}(\ln \epsilon + \pi)}{4 - \epsilon^2} \cdot \pi \epsilon \rightarrow 0$$

as $\epsilon \rightarrow 0$. Next, we parametrize C_{left} by $z(t) = e^{i\pi}t$ where $t : R \rightarrow \epsilon$. The corresponding integral is

$$\begin{aligned} \int_{C_{\text{left}}} \frac{\sqrt{z} \text{Log}(z)}{z^2 + 4} dz &= \int_R^\epsilon \frac{\sqrt{(te^{i\pi})} \text{Log}(te^{i\pi})}{(te^{i\pi})^2 + 4} e^{i\pi} dt \\ &= \int_\epsilon^R \frac{e^{i\pi/2} \sqrt{t} (\ln t + i\pi)}{t^2 + 4} dt \\ &= \int_\epsilon^R i \frac{\sqrt{t} \ln t}{t^2 + 4} dt - \pi \frac{\sqrt{t}}{t^2 + 4} dt \end{aligned}$$

The loop integral is thus

$$i \int_0^\infty \frac{\sqrt{t} \ln t}{t^2 + 4} dt - \pi \frac{\sqrt{t}}{t^2 + 4} dt + \int_0^\infty \frac{\sqrt{x} \ln x}{x^2 + 4} dx = \frac{\pi}{2} \left(\ln 2 - \frac{\pi}{2} + i \left(\frac{\pi}{2} + \ln 2 \right) \right)$$

Taking the imaginary part of this equation gives the final answer:

$$\int_0^\infty \frac{\sqrt{t} \ln t}{t^2 + 4} dt = \frac{\pi}{2} \left(\frac{\pi}{2} + \ln 2 \right).$$