

Mathematics 305 — Final Exam

Practice Final A

- The test consists of 12 pages and 11 questions worth a total of 60 marks.
- Electronic devices of any kind (including calculators, cell phones, etc.) are forbidden.
- No work on this page will be marked.
- If you require extra paper then put up your hand and ask your instructor. You must indicate your name, student number, test number and question number on any extra pages.

Student number								
Section								
Name								
Signature								

1. (a) 4 marks Find all solutions to $e^{z^2} = -1$.

(b) 4 marks Find all solutions to $\sqrt{z} = 1 - i$, where $\sqrt{z}$ denotes the principal branch.

2. 6 marks Consider the branch

$$\sqrt{z^2 - 1} = r_1^{1/2} r_2^{1/2} e^{i(\theta_1 + \theta_2)/2}, \quad 0 \leq \theta_1 < 2\pi, \quad -\pi < \theta_2 \leq \pi,$$

with $z - 1 = r_1 e^{i\theta_1}$ and $z + 1 = r_2 e^{i\theta_2}$, and define the inverse cosine function by

$$\cos^{-1}(z) = -i \operatorname{Log} \left(z + \sqrt{z^2 - 1} \right),$$

where Log denotes the principal branch of the logarithm. Using this definition, compute $\cos^{-1}(0)$ and find the image of

$$S = \{-1 \leq \operatorname{Re}(z) \leq 1, \quad \operatorname{Im}(z) = 0\}.$$

In other words, find $S' = \cos^{-1}(S)$.

3. 5 marks Let $f(z)$ be an analytic function. Suppose

$$|f(e^{i\theta})| \leq 3$$

for all angles $0 \leq \theta \leq 2\pi$. Further suppose $f(z)$ has a zero of order 1 at $z = 0$. Find the largest possible value of $|f(1/2)|$. Hint: consider $g(z) = f(z)/z$.

4. 4 marks Evaluate

$$\int_C \frac{1+z}{z^2} dz$$

where C is the upper semi-circle $C = \{|z| = 1, \operatorname{Im} z \geq 0\}$ running clockwise from $z = -1$ to $z = 1$.

5. 4 marks Compute $\int_C z^2 \sin \frac{1}{z} dz$ where C is the counterclockwise unit circle $|z| = 1$.

6. 4 marks Evaluate $\int_C \frac{1}{z^3(z+6)} dz$ where C is the counterclockwise circle $|z| = 2$.

7. 6 marks Find the first three terms of the Laurent series centred at $z_0 = 3$ of

$$f(z) = \frac{1}{(z-1)^2(z-3)}$$

valid in $0 < |z-3| < 2$.

8. 5 marks Find the number of roots, counted with multiplicity, of

$$f(z) = 2z^5 + 5z^3 + z + 1$$

in the region $\{1 < |z| < 2\}$.

9. 6 marks Calculate

$$\int_0^\pi \frac{2}{5 + 2 \cos \theta} d\theta$$

by complex contour integration.

10. 6 marks Calculate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 2x + 2} dx$$

by complex contour integration.

11. 6 marks Calculate

$$\int_0^{\infty} \frac{\sqrt{x} \ln x}{x^2 + 4} dx$$

by complex contour integration.