

Mathematics 305 — Final Exam

Practice Final B

- The test consists of 12 pages and 11 questions worth a total of 60 marks.
- Electronic devices of any kind (including calculators, cell phones, etc.) are forbidden.
- No work on this page will be marked.
- If you require extra paper then put up your hand and ask your instructor. You must indicate your name, student number, test number and question number on any extra pages.

Student number								
Section								
Name								
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1. (a) 4 marks Find all solutions to

$$\operatorname{Log}(1/z) = -\operatorname{Log}(z)$$

where Log denotes the principal branch of the logarithm.

Solution: Write $z = |z|e^{i\varphi}$ with $-\pi < \varphi \leq \pi$. Then

$$\operatorname{Log}(z) = \ln |z| + i\varphi.$$

Since $1/z = |z|^{-1}e^{-i\varphi}$, if $-\pi < \varphi < \pi$, then

$$\operatorname{Log}(1/z) = -\ln |z| - i\varphi.$$

But this is not true if $\varphi = \pi$. For example

$$\operatorname{Log}(-1) = i\pi, \quad \operatorname{Log}(1/(-1)) = i\pi.$$

The set of all solutions is:

$$z = |z|e^{i\varphi}, \quad -\pi < \varphi < \pi, \quad |z| \neq 0.$$

In other words, $z \in \mathbb{C} \setminus (-\infty, 0]$.

- (b) 4 marks Find all solutions to $\sin z + \cos z = 1$.

Solution: We rewrite this equation using the complex exponential function.

$$\frac{1}{2i}(e^{iz} - e^{-iz}) + \frac{1}{2}(e^{iz} + e^{-iz}) = 1.$$

Multiplying by $2i$ and regrouping gives

$$(1+i)e^{iz} + (i-1)e^{-iz} = 2i.$$

Multiplying by e^{iz} and putting everything to one side gives

$$(1+i)e^{2iz} - 2ie^{iz} + (i-1) = 0.$$

Using the quadratic formula we can solve for $x = e^{iz}$.

$$e^{iz} = \frac{2i \pm (-4 - 4(1+i)(i-1))^{1/2}}{2(1+i)}.$$

This simplifies to

$$e^{iz} = \frac{i \pm 1}{(1+i)} \frac{1-i}{1-i} = \frac{i \pm 1}{2}(1-i).$$

The two options are

$$e^{iz} = 1, \quad e^{iz} = i = e^{i\pi/2}.$$

Therefore

$$z = 2\pi k, \quad \text{or} \quad z = \frac{\pi}{2} + 2\pi k$$

for $k = 0, \pm 1, \pm 2, \dots$

2. 6 marks Construct a branch of $f(z) = (z^2 + 1)^{1/3}$ such that $f(z)$ is analytic in $|z| < 1$ and for which $f(0) = 1$. Evaluate $f(1 + i)$.

Solution: First, we factor the polynomial.

$$f(z) = (z + i)^{1/3}(z - i)^{1/3}.$$

Next, we setup our angles.

$$z - i = r_1 e^{i\varphi_1}, \quad z + i = r_2 e^{i\varphi_2},$$

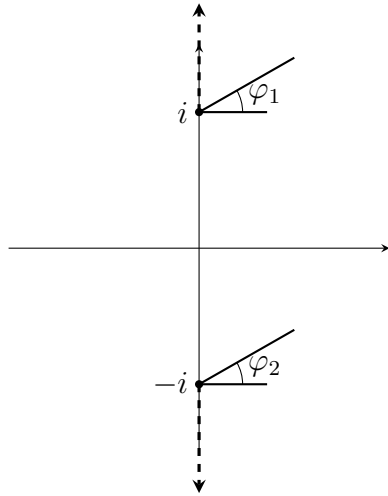
where

$$-\frac{3\pi}{2} < \varphi_1 \leq \frac{\pi}{2}, \quad -\frac{\pi}{2} < \varphi_2 \leq \frac{3\pi}{2}.$$

The function is then

$$f(z) = (r_1 r_2)^{1/3} e^{i\varphi_1/3} e^{i\varphi_2/3}.$$

See the attached diagram.



The angle setup is such that the function can only potentially jump across either $i[1, \infty)$ or $i[-\infty, -1]$, but this construction is analytic inside $|z| < 1$. So there is no need to check continuity across any of the interfaces.

There are three choices of branch for the cubed root, so we should check that we are on the right branch. For this, we verify $f(0) = 1$.

$$f(0) = ((1)(1))^{1/3} e^{i(-\pi/2)/3} e^{i(\pi/2)/3} = 1,$$

and so we are on the right branch. We can now compute $f(1 + i)$. First, we should compute the polar coordinates (r_1, φ_1) and (r_2, φ_2) for the point $1 + i$.

$$r_1 = 1, \quad \varphi_1 = 0, \quad r_2 = \sqrt{5}, \quad \varphi_2 = \arctan(2).$$

Substituting this into the formula for $f(z)$ gives

$$f(1 + i) = (\sqrt{5})^{1/3} e^{i\arctan(2)/3}.$$

3. 5 marks Find the image $S' = f(S)$ of the unit circle $S = \{|z| = 1\}$ under the mapping

$$f(z) = z - \frac{1}{z}.$$

Sketch this set S' .

Solution: Write an element $z \in S$ as $z = e^{i\varphi}$ with $-\pi < \varphi \leq \pi$. Write $w = f(z)$. Then

$$w = e^{i\varphi} - e^{-i\varphi}.$$

Using the definition of sine, this is

$$w = 2i \sin \varphi.$$

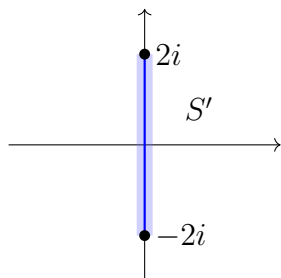
If $-\pi < \varphi \leq \pi$, then $-1 \leq \sin \varphi \leq 1$. It follows that

$$w = iy, \quad -2 \leq y \leq 2.$$

In other words,

$$S' = \{iy : -2 \leq y \leq 2\}.$$

Here is a sketch:



4. 4 marks Evaluate

$$\int_{|z|=1} z^3 \cos(z^{-1}) dz$$

where the contour runs counterclockwise.

Solution: The Laurent expansion of the integrand is

$$\begin{aligned} z^3 \cos z^{-1} &= z^3 \left[1 - \frac{1}{2!} z^{-2} + \frac{1}{4!} z^{-4} - \frac{1}{6!} z^{-6} + \cdots \right] \\ &= z^3 - \frac{1}{2} z + \frac{1}{4!} z^{-1} - \frac{1}{6!} z^{-3} + \cdots \end{aligned}$$

Therefore

$$\operatorname{Res}(z^3 \cos z^{-1}, 0) = \frac{1}{4!}.$$

Since the unit circle encloses the only singularity of $z^3 \cos z^{-1}$, by the residue theorem we conclude

$$\int_{|z|=1} z^3 \cos(z^{-1}) dz = \frac{(2\pi i)}{4!} = \frac{i\pi}{12}.$$

5. 4 marks Evaluate

$$\int_C \frac{z}{\operatorname{Log}(z)} dz$$

where $C = \{|z - 1| = \frac{1}{2}\}$ runs counterclockwise.

Solution: The integrand has a singularity inside C at $z = 1$. Therefore

$$\int_C \frac{z}{\operatorname{Log}(z)} dz = (2\pi i) \operatorname{Res}\left(\frac{z}{\operatorname{Log}(z)}, 1\right).$$

Let $P = z$ and $Q = \operatorname{Log}(z)$. Then

$$\operatorname{Res}\left(\frac{z}{\operatorname{Log}(z)}, 1\right) = \frac{P(1)}{Q'(1)} = \frac{1}{1/1} = 1.$$

Therefore

$$\int_C \frac{z}{\operatorname{Log}(z)} dz = 2\pi i.$$

6. 4 marks Evaluate

$$\int_C \frac{e^z}{(z-1)^5} dz$$

where $C = \{|z-1| = \frac{1}{2}\}$ runs counterclockwise.

Solution: The integrand has a singularity at $z = 1$ which is inside C . Let $g(z) = e^z$. By Cauchy's derivative formula,

$$\int_C \frac{g(z)}{(z-1)^5} dz = \frac{(2\pi i)}{4!} g^{(4)}(1).$$

Since $g^{(4)}(1) = e$, we have

$$\int_C \frac{g(z)}{(z-1)^5} dz = i \frac{e\pi}{12}.$$

7. 6 marks Let $n \in \mathbb{Z}$. Compute the Laurent expansion at $z_0 = 0$ of

$$\frac{z^n}{z^2 - 1}$$

valid in the region $|z| > 1$. Then compute

$$\int_{|z|=5} \frac{z^n}{z^2 - 1} dz$$

where the contour runs counterclockwise.

Solution: We will use partial fractions expansion on

$$\frac{z^n}{z^2 - 1} = z^n \left[\frac{1}{(z - 1)(z + 1)} \right].$$

For this, write

$$\frac{1}{(z - 1)(z + 1)} = \frac{A}{z - 1} + \frac{B}{z + 1} \implies 1 = A(z + 1) + B(z - 1).$$

Substituting $z = 1$ and $z = -1$ leads to that $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. Therefore

$$\frac{z^n}{z^2 - 1} = z^n \left[\frac{1}{2} \frac{1}{z - 1} - \frac{1}{2} \frac{1}{z + 1} \right].$$

We would like to use a geometric series expansion valid on $|z| > 1$. So we must bring out z^{-1} as the expansion parameter.

$$\frac{z^n}{z^2 - 1} = z^n \left[\frac{1}{2z} \frac{1}{1 - z^{-1}} - \frac{1}{2z} \frac{1}{1 + z^{-1}} \right].$$

Using the geometric series formula, we obtain

$$\frac{z^n}{z^2 - 1} = z^n \left[\frac{1}{2z} \left[\sum_{k=0}^{\infty} z^{-k} \right] - \frac{1}{2z} \left[\sum_{k=0}^{\infty} (-1)^k z^{-k} \right] \right].$$

We see that all the even k cancel themselves and odd k combine. We are left with

$$\frac{z^n}{z^2 - 1} = z^n \left[\frac{1}{z} \left[\sum_{k=\text{odd}, k \geq 1} z^{-k} \right] \right].$$

Thus

$$\frac{z^n}{z^2 - 1} = z^{n-1} \sum_{k=\text{odd}, k \geq 1} z^{-k}.$$

For the integral, recall that

$$\int_C z^k dz = \begin{cases} 0, & k \neq -1, \\ 2\pi i, & k = -1. \end{cases}$$

So for a non-zero integral, we need to have $k = n$ appear in the summation so that $z^{n-1}z^{-k} = z^{-1}$, which does not always happen; the integer n will only be attained by a k in the sum if it is odd and $n \geq 1$. Therefore

$$\int_{|z|=5} \frac{z^n}{z^2 - 1} dz = \begin{cases} 2\pi i, & n \text{ odd and } n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

8. 5 marks Find the number of zeros of the polynomial

$$f(z) = z^4 + z^3 + 5z^2 + 2z + 4$$

in the right half-plane $\operatorname{Re}(z) > 0$.

Solution: By the Nyquist theorem, the number N of such zeros is given by

$$N = \frac{1}{2\pi} \left(4\pi + 2\Delta_{\Gamma_+} \arg f \right).$$

So we must compute the change in argument of $f(z)$ along the straight line ray Γ_+ . Let

$$f(iy) = (iy)^4 + (iy)^3 + 5(iy)^2 + 2(iy) + 4$$

for $y \geq 0$. This simplifies to

$$f(iy) = P_R(y) + iP_I(y)$$

with

$$P_R(y) = y^4 - 5y^2 + 4, \quad P_I(y) = -y^3 + 2y.$$

To form a sketch, we use the following information:

- As $y \rightarrow \infty$, we note that $P_R > 0$, $P_I < 0$ and

$$\arctan \frac{|P_I|}{|P_R|} \rightarrow 0.$$

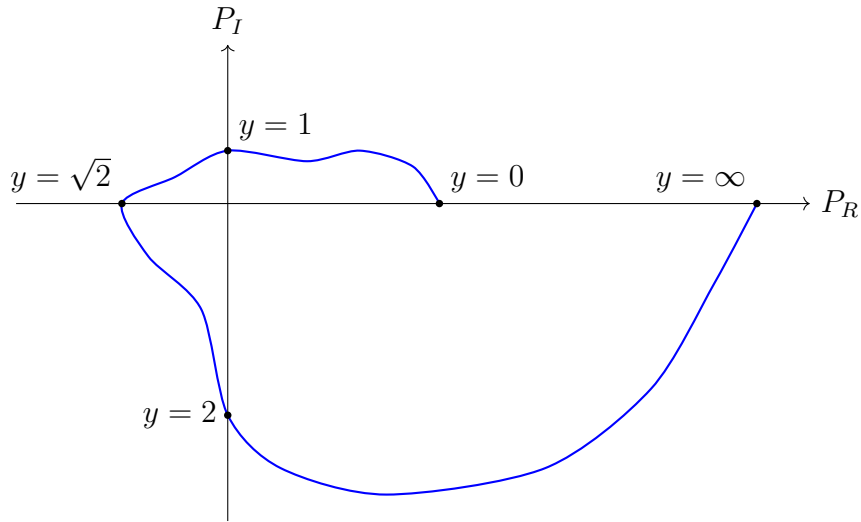
- $P_I = 0$. This equation is solved for $y \geq 0$ when

$$y(-y^2 + 2) = 0 \implies y = 0, \quad y = \sqrt{2}.$$

- $P_R = 0$. This equation is solved for $y \geq 0$ when

$$y^4 - 5y^2 + 4 = 0 \implies y^2 = \frac{5 \pm \sqrt{25 - 4(4)}}{2} = \frac{5 \pm 3}{2}.$$

Solutions are $y = 2$ and $y = 1$.



From the sketch, we see that $\Delta_{\Gamma_+} \arg f = -2\pi$. The minus sign is included since the curve from $y = \infty$ to $y = 0$ travels in the clockwise direction. Therefore

$$N = \frac{1}{2\pi} \left(4\pi - 2(2\pi) \right) = 0.$$

9. 6 marks Calculate

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + 2x + 2} dx$$

by complex contour integration.

Solution: We complexify the integrand and consider the analytic function

$$f(z) = \frac{P}{Q} = \frac{\sqrt{z}}{z^2 + 2z + 2}.$$

Here $\sqrt{z} = \sqrt{|z|}e^{i\varphi/2}$ when $z = |z|e^{i\varphi}$ with $0 \leq \varphi < 2\pi$; this is not the principal branch. The poles are at

$$z = \frac{-2 \pm \sqrt{4 - 4(2)}}{2}$$

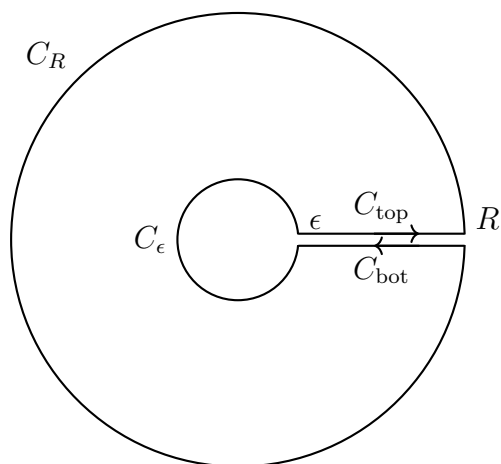
which simplifies to

$$z = -1 + i, \quad \text{or} \quad z = -1 - i.$$

Consider the closed loop

$$C = C_\epsilon + C_{\text{top}} + C_R + C_{\text{bot}}$$

where C_ϵ runs along $|z| = \epsilon$, C_{top} runs along the x-axis from $+\epsilon$ to $+R$, C_R runs along $|z| = R$, and C_{bot} runs from $+R$ back to $+\epsilon$.



By the residue theorem

$$\begin{aligned}
\int_C \frac{\sqrt{z}}{z^2 + 2z + 2} dz &= (2\pi i) \left[\text{Res}(f, -1 + i) + \text{Res}(f, -1 - i) \right] \\
&= (2\pi i) \left[\frac{P(-1 + i)}{Q'(-1 + i)} + \frac{P(-1 - i)}{Q'(-1 - i)} \right] \\
&= (2\pi i) \left[\frac{2^{1/4} e^{i3\pi/8}}{2(-1 + i) + 2} + \frac{2^{1/4} e^{i5\pi/8}}{2(-1 - i) + 2} \right] \\
&= \pi 2^{1/4} (e^{i3\pi/8} - e^{i5\pi/8})
\end{aligned}$$

The integral over C_R goes to zero:

$$\left| \int_{C_R} \frac{\sqrt{z}}{z^2 + 2z + 2} dz \right| \leq \frac{R^{1/2}}{R^2 - 2R - 2} \cdot 2\pi R \rightarrow 0$$

as $R \rightarrow \infty$. The integral over C_ϵ also goes to zero:

$$\left| \int_{C_\epsilon} \frac{\sqrt{z}}{z^2 + 2z + 2} dz \right| \leq \frac{\epsilon^{1/2}}{2 - \epsilon - \epsilon^2} \cdot 2\pi\epsilon \rightarrow 0$$

as $\epsilon \rightarrow 0$. Next, we parametrize C_{bot} by $z(t) = e^{i2\pi}t$ where $t : R \rightarrow \epsilon$. The corresponding integral is

$$\begin{aligned}
\int_{C_{\text{bot}}} \frac{\sqrt{z}}{z^2 + 2z + 2} dz &= \int_R^\epsilon \frac{\sqrt{(te^{i2\pi})}}{t^2 + 2t + 2} dt \\
&= \int_R^\epsilon \frac{e^{i\pi} \sqrt{t}}{t^2 + 2t + 2} dt \\
&\rightarrow \int_0^\infty \frac{\sqrt{t}}{t^2 + 2t + 2} dt.
\end{aligned}$$

The loop integral is thus

$$0 + \int_0^\infty \frac{\sqrt{t}}{t^2 + 2t + 2} dt + 0 + \int_0^\infty \frac{\sqrt{t}}{t^2 + 2t + 2} dt = \pi 2^{1/4} (e^{i3\pi/8} - e^{i5\pi/8}).$$

Therefore

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + 2x + 2} dx = \frac{\pi}{2} 2^{1/4} (e^{i3\pi/8} - e^{i5\pi/8}).$$

We can simplify the right-hand side using

$$e^{i3\pi/8} - e^{i5\pi/8} = e^{i\pi/2} (e^{-i\pi/8} - e^{i\pi/8}) = 2 \sin\left(\frac{\pi}{8}\right).$$

Therefore

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + 2x + 2} dx = \pi 2^{1/4} \sin\left(\frac{\pi}{8}\right).$$

10. 6 marks The following integral can be computed as

$$I = \int_0^\pi \frac{\cos(2\theta)}{2 + \cos \theta} d\theta = 2\pi i \left(\text{Res}(f, z_0) + \text{Res}(f, z_1) \right).$$

What are $f(z)$, z_0 , and z_1 ?

Solution: Let $z = e^{i\varphi}$ and $dz = ie^{i\varphi}d\varphi$. Then

$$\cos \varphi = \frac{1}{2}(z + z^{-1}).$$

The integral becomes

$$I = \frac{1}{2} \int_{|z|=1} \frac{\frac{1}{2}(z^2 + z^{-2})}{2 + \frac{1}{2}(z + z^{-1})} \frac{dz}{iz}.$$

This simplifies to

$$I = \frac{1}{2} \int_{|z|=1} \frac{(z^4 + 1)}{4z^2 + z^3 + z} \frac{dz}{iz}.$$

Hence

$$I = \int_{|z|=1} -\frac{i}{2} \frac{(z^4 + 1)}{z^4 + 4z^3 + z^2} dz.$$

We read-off

$$f(z) = -\frac{i}{2} \frac{z^4 + 1}{z^4 + 4z^3 + z^2}.$$

To find z_0 and z_1 , we solve

$$z^4 + 4z^3 + z^2 = 0 \implies z^2(z^2 + 4z + 1) = 0.$$

Either $z = 0$ or

$$z = \frac{-4 \pm \sqrt{16 - 4}}{2} = -2 \pm \sqrt{3}.$$

Only $-2 + \sqrt{3}$ is inside the unit circle. Therefore

$$z_0 = 0, \quad z_1 = -2 + \sqrt{3}.$$

11. 6 marks Calculate

$$\int_0^\infty \frac{\ln(x)}{x^2 + 4} dx$$

by complex contour integration.

Solution:

Consider

$$f(z) = \frac{P}{Q} = \frac{\text{Log } z}{z^2 + 4}.$$

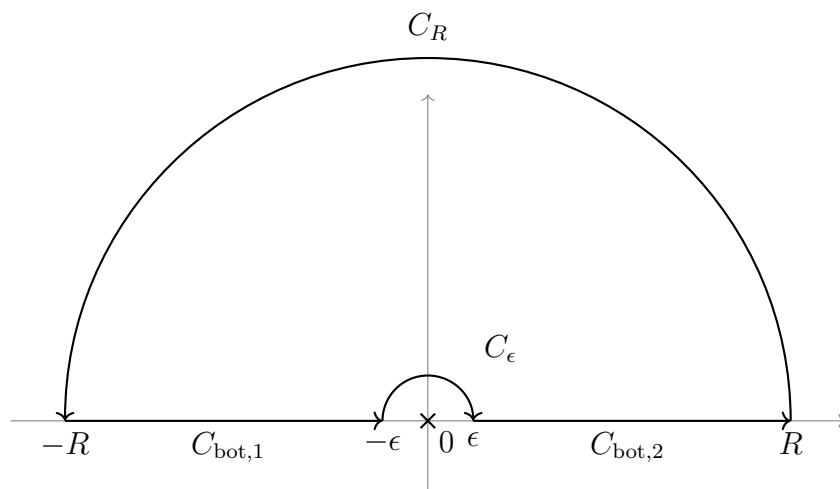
Here we replaced $\ln(x)$ with the principal logarithm $\text{Log } z$. The function $f(z)$ is undefined at $z = 0$ and there are poles at

$$z = \pm 2i.$$

Consider the contour

$$C = C_R + C_{\text{bot},1} + C_\epsilon + C_{\text{bot},2},$$

as shown schematically below.



The only singularity of f enclosed in C is $+2i$. By the residue theorem

$$\begin{aligned} \int_C \frac{\text{Log } z}{z^2 + 4} dz &= (2\pi i) \text{Res}(f, 2i) \\ &= (2\pi i) \frac{P(2i)}{Q'(2i)} \\ &= (2\pi i) \frac{\text{Log}(2i)}{2(2i)} \\ &= \frac{\pi}{2} \left(\ln 2 + i \frac{\pi}{2} \right). \end{aligned}$$

We now consider each piece of the curve C . First, note that the integral over C_R tends to zero:

$$\left| \int_{C_R} \frac{\text{Log } z}{z^2 + 4} dz \right| \leq \left(\frac{\ln R + \pi}{R^2 - 4} \right) \cdot \pi R \rightarrow 0$$

as $R \rightarrow \infty$. Next, we estimate the integral over C_ϵ .

$$\left| \int_{C_\epsilon} \frac{\text{Log } z}{z^2 + 4} dz \right| \leq \left(\frac{|\ln \epsilon| + \pi}{4 - \epsilon^2} \right) \cdot \pi \epsilon \rightarrow 0$$

as $\epsilon \rightarrow 0$. Next, we consider the integral over $C_{\text{bot},1}$. For this, parametrize $z(t) = e^{i\pi}t$ from $t : R \rightarrow \epsilon$. The result is

$$\begin{aligned} \int_{C_{\text{bot},1}} \frac{\text{Log } z}{z^2 + 4} dz &= \int_R^\epsilon \frac{\text{Log}(te^{i\pi})}{t^2 + 4} e^{i\pi} dt \\ &= \int_\epsilon^R \frac{\ln t + i\pi}{t^2 + 4} dt \\ &\rightarrow \int_0^\infty \frac{\ln x + i\pi}{x^2 + 4} dx \end{aligned}$$

as $R \rightarrow \infty$ and $\epsilon \rightarrow 0$. Therefore the loop integral equation is

$$\int_0^\infty \frac{\ln x + i\pi}{x^2 + 4} dx + \int_0^\infty \frac{\ln x}{x^2 + 4} dx = \frac{\pi}{2} \left(\ln 2 + i\frac{\pi}{2} \right).$$

Taking the real part of this equation yields

$$\int_0^\infty \frac{\ln x}{x^2 + 4} dx = \frac{\pi}{4} \ln 2.$$