

Mathematics 305 — Final Exam

Practice Final B

- The test consists of 12 pages and 11 questions worth a total of 60 marks.
- Electronic devices of any kind (including calculators, cell phones, etc.) are forbidden.
- No work on this page will be marked.
- If you require extra paper then put up your hand and ask your instructor. You must indicate your name, student number, test number and question number on any extra pages.

Student number								
Section								
Name								
Signature								

1. (a) 4 marks Find all solutions to

$$\operatorname{Log}(1/z) = -\operatorname{Log}(z)$$

where Log denotes the principal branch of the logarithm.

- (b) 4 marks Find all solutions to $\sin z + \cos z = 1$.

2. 6 marks Construct a branch of $f(z) = (z^2 + 1)^{1/3}$ such that $f(z)$ is analytic in $|z| < 1$ and for which $f(0) = 1$. Evaluate $f(1 + i)$.

3. 5 marks Find the image $S' = f(S)$ of the unit circle $S = \{|z| = 1\}$ under the mapping

$$f(z) = z - \frac{1}{z}.$$

Sketch this set S' .

4. 4 marks Evaluate

$$\int_{|z|=1} z^3 \cos(z^{-1}) dz$$

where the contour runs counterclockwise.

5. 4 marks Evaluate

$$\int_C \frac{z}{\operatorname{Log}(z)} dz$$

where $C = \{|z - 1| = \frac{1}{2}\}$ runs counterclockwise.

6. 4 marks Evaluate

$$\int_C \frac{e^z}{(z-1)^5} dz$$

where $C = \{|z-1| = \frac{1}{2}\}$ runs counterclockwise.

7. 6 marks Let $n \in \mathbb{Z}$. Compute the Laurent expansion at $z_0 = 0$ of

$$\frac{z^n}{z^2 - 1}$$

valid in the region $|z| > 1$. Then compute

$$\int_{|z|=5} \frac{z^n}{z^2 - 1} dz$$

where the contour runs counterclockwise.

8. 5 marks Find the number of zeros of the polynomial

$$f(z) = z^4 + z^3 + 5z^2 + 2z + 4$$

in the right half-plane $\operatorname{Re}(z) > 0$.

9. 6 marks Calculate

$$\int_0^{\infty} \frac{\sqrt{x}}{x^2 + 2x + 2} dx$$

by complex contour integration.

10. 6 marks The following integral can be computed as

$$I = \int_0^\pi \frac{\cos(2\theta)}{2 + \cos \theta} d\theta = 2\pi i \left(\operatorname{Res}(f, z_0) + \operatorname{Res}(f, z_1) \right).$$

What are $f(z)$, z_0 , and z_1 ?

11. 6 marks Calculate

$$\int_0^\infty \frac{\ln(x)}{x^2 + 4} dx$$

by complex contour integration.