

Mathematics 305 — Final Exam

Practice Final D

- The test consists of 17 pages and 11 questions worth a total of 60 marks.
- Electronic devices of any kind (including calculators, cell phones, etc.) are forbidden.
- No work on this page will be marked.
- If you require extra paper then put up your hand and ask your instructor. You must indicate your name, student number, test number and question number on any extra pages.

Student number								
Section								
Name								
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1. (a) 4 marks Simplify

$$\cos[i \log(-1 + i)]$$

into the form $a + ib$.

Solution: Since

$$\cos X = \frac{1}{2}(e^{iX} + e^{-iX}),$$

we must compute

$$\cos[i \log(-1 + i)] = \frac{1}{2} \left(\exp \left(-\log(-1 + i) \right) + \exp \left(\log(-1 + i) \right) \right).$$

Since $\exp \log X = X$, this becomes

$$\cos[i \log(-1 + i)] = \frac{1}{2} \left(\frac{1}{-1 + i} - 1 + i \right).$$

We can simplify

$$\frac{1}{-1 + i} = \frac{1}{-1 + i} \cdot \frac{-1 - i}{-1 - i} = \frac{-1 - i}{2}.$$

Therefore

$$\cos[i \log(-1 + i)] = \frac{1}{2} \left(\frac{-1 - i}{2} + -1 + i \right).$$

The final answer is then

$$\cos[i \log(-1 + i)] = -\frac{3}{4} + \frac{i}{4}.$$

- (b) 4 marks Simplify

$$\text{Log} \left(\frac{\sqrt{3} + i}{\sqrt{2}(1 + i)} \right)$$

into the form $a + ib$.

Solution: First, we rewrite the numerator in polar coordinates.

$$\sqrt{3} + i = 2e^{i\pi/6}.$$

Second, we rewrite the denominator in polar coordinates.

$$\sqrt{2}(1 + i) = 2e^{i\pi/4}.$$

Therefore

$$\frac{\sqrt{3} + i}{\sqrt{2}(1 + i)} = \frac{e^{i\pi/6}}{e^{i\pi/4}} = e^{-i\pi/12}.$$

The principal logarithm evaluates to

$$\operatorname{Log}\left(\frac{\sqrt{3}+i}{\sqrt{2}(1+i)}\right) = -\frac{i\pi}{12}.$$

2. 6 marks Find a branch of $(z - 1)^{-1/2}$ that is analytic on $\mathbb{C} \setminus S$, where

$$S = \{z : \operatorname{Re}(z) = 1, \operatorname{Im}(z) \leq 0\}.$$

Using this branch, compute

$$\int_C (z - 1)^{-1/2} dz$$

where C is the semicircle centered at 1 going counterclockwise from 2 to 0.

Solution: We start by setting up our angle. Define polar coordinates based at 1 by setting

$$z - 1 = re^{i\varphi},$$

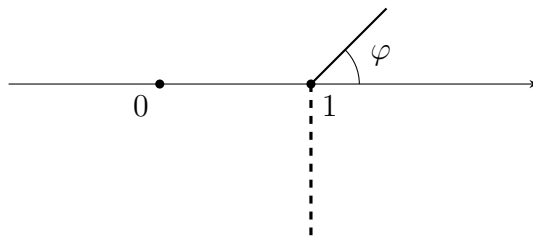
where

$$-\frac{\pi}{2} < \varphi \leq \frac{3\pi}{2}.$$

The analytic function on $\mathbb{C} \setminus S$ is then

$$f(z) = \frac{e^{-i\varphi/2}}{\sqrt{r}}.$$

See the attached diagram.



We now compute the line integral. Parametrize C by

$$z(t) = 1 + e^{it}, \quad 0 \leq t \leq \pi.$$

Then $dz = ie^{it}dt$ and

$$f(z(t)) = e^{-it/2}.$$

The integral is

$$\int_C (z - 1)^{-1/2} dz = \int_0^\pi e^{-it/2} (ie^{it} dt) = i \int_0^\pi e^{it/2} dt.$$

This evaluates to

$$\int_C (z - 1)^{-1/2} dz = 2 \Big|_0^\pi e^{it/2} = 2(i - 1).$$

3. 5 marks Express the residue at 0 of

$$\frac{e^{1/z}}{1-z}$$

as an infinite sum, and calculate this sum.

Solution: We expand

$$\begin{aligned}\frac{e^{1/z}}{1-z} &= \left[\sum_{k=0}^{\infty} \frac{z^{-k}}{k!} \right] \left[\sum_{n=0}^{\infty} z^n \right] \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{z^{n-k}}{k!}.\end{aligned}$$

The residue is the coefficient of z^{-1} , which occurs when $n - k = -1$, namely $k = n + 1$.

$$\text{Res}\left(\frac{e^{1/z}}{1-z}, 0\right) = \sum_{k=0}^{\infty} \frac{1}{(n+1)!}.$$

This sum is

$$\sum_{k=0}^{\infty} \frac{1}{(n+1)!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e - 1.$$

Therefore

$$\text{Res}\left(\frac{e^{1/z}}{1-z}, 0\right) = e - 1.$$

4. 4 marks Compute

$$\int_C |z|^2 dz$$

where C is the directed line segment from i to $1 + i$.

Solution: We parametrize C by

$$z(t) = (1 - t)i + t(1 + i), \quad 0 \leq t \leq 1.$$

This simplifies to

$$z = t + i, \quad dz = dt.$$

Therefore

$$\int_C |z|^2 dz = \int_0^1 |t + i|^2 dt.$$

We evaluate this integral to

$$\int_C |z|^2 dz = \int_0^1 (t^2 + 1) dt = \left|_0^1 \left[\frac{t^3}{3} + t \right] = \frac{1}{3} + 1.$$

Therefore

$$\int_C |z|^2 dz = \frac{4}{3}.$$

5. 4 marks Compute

$$\int_C \frac{100! e^{iz}}{(z+1)^{100}} dz$$

where $C = \{|z| = 5\}$ is a counterclockwise circle.

Solution: Cauchy's derivative formula is

$$\int_C \frac{f(z)}{(z+1)^{100}} dz = (2\pi i) \frac{f^{(99)}(-1)}{99!}.$$

For this problem, we let $f(z) = 100! e^{iz}$. Then

$$\int_C \frac{100! e^{iz}}{(z+1)^{100}} dz = \frac{2\pi i}{99!} 100! i^{99} e^{-i}.$$

Since $i^{100} = 1$, we have $i^{99} = -i$. Therefore

$$\int_C \frac{100! e^{iz}}{(z+1)^{100}} dz = (200\pi)(\cos 1 - i \sin 1).$$

6. 4 marks Compute

$$\int_C \frac{z \cos z}{(2z - \pi)^2} dz$$

where $C = \{|z| = 4\}$ is a counterclockwise circle.

Solution: The singularity of the integrand occurs at

$$z = \frac{\pi}{2}$$

which is within C . By Cauchy's derivative formula,

$$\int_C \frac{f(z)}{(z - \frac{\pi}{2})^2} dz = (2\pi i) f'(\pi/2).$$

For this particular problem, we take $f(z) = \frac{z \cos z}{4}$. Its derivative is

$$f'(z) = \frac{1}{4} \left[\cos z - z \sin z \right].$$

Therefore $f'(\pi/2) = -\pi/8$. We conclude

$$\int_C \frac{f(z)}{(z - \frac{\pi}{2})^2} dz = (2\pi i) \left(-\frac{\pi}{8} \right)$$

and hence

$$\int_C \frac{z \cos z}{(2z - \pi)^2} dz = -\frac{\pi^2 i}{4}.$$

7. 6 marks Find the Laurent series of

$$f(z) = \frac{z}{z^2 - z - 2}$$

centered at $z_0 = 0$ that converges for $|z| > 2$. Explicitly calculate the first two terms, then compute $\int_{|z|=4} f(z) dz$ where the circle $|z| = 4$ is oriented counterclockwise.

Solution: We factor the function and search for its partial fractions decomposition:

$$f = \frac{z}{(z-2)(z+1)} = \frac{A}{z-2} + \frac{B}{z+1}.$$

The coefficients A, B are found by multiplying both sides by $(z-2)(z+1)$, which leads to

$$z = A(z+1) + B(z-2).$$

Substituting $z = 2$ gives $A = 2/3$ and substituting $z = -1$ gives $B = 1/3$. Therefore

$$f(z) = \frac{2}{3(z-2)} + \frac{1}{3(z+1)} = \frac{2}{3z} \frac{1}{1-2z^{-1}} + \frac{1}{3z} \frac{1}{1+z^{-1}}.$$

By the formula for a geometric series, we obtain

$$f(z) = \frac{2}{3z} \left[\sum_{n=0}^{\infty} (2/z)^n \right] + \frac{1}{3z} \left[\sum_{n=0}^{\infty} (-1)^n z^{-n} \right].$$

The first two terms are

$$f(z) = \frac{2}{3z} \left[1 + \frac{2}{z} \right] + \frac{1}{3z} \left[1 - \frac{1}{z} \right] + \dots$$

which becomes

$$f(z) = z^{-1} + z^{-2} + \dots$$

This Laurent expansion holds for $|z| > 2$, and hence holds along the circle $|z| = 4$. Therefore to integrate $f(z)$, we can integrate each term in the series. Since

$$\int_{|z|=4} z^m dz = \begin{cases} 0 & \text{if } m \neq -1, \\ 2\pi i & \text{if } m = -1, \end{cases}$$

we conclude

$$\int_{|z|=4} f(z) dz = 2\pi i.$$

8. 5 marks Find the number of zeros, counted with multiplicity, of

$$f(z) = z^5 + 3z + 1$$

in the disk $\{|z| < 2\}$.

Solution: We note that on $|z| = 2$, we have

$$|3z + 1| < |z^5|.$$

Indeed,

$$|3z + 1| \leq 3|z| + 1 = 3 \cdot 2 + 1 = 7$$

while

$$|z^5| = |z|^5 = 2^5 = 32$$

on $|z| = 2$. By Rouché's theorem, $z^5 + 3z + 1$ and z^5 have the same number of zeros inside $|z| < 2$ when counted with multiplicity. Since z^5 has a zero of order 5, we conclude that $f(z)$ has five zeros when counted with multiplicity.

9. 6 marks Calculate

$$\int_{-\infty}^{\infty} \frac{1}{x^4 + 1} dx$$

by complex contour integration.

Solution:

We complexify the integrand by letting

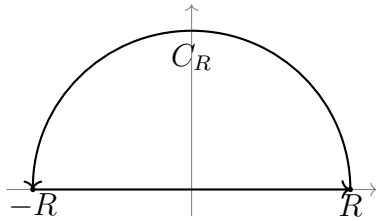
$$f(z) = \frac{1}{z^4 + 1},$$

The singularities are located at

$$z = e^{i\pi/4}\omega_4^k, \quad \omega_4 = e^{i\pi/2}, \quad k = 0, 1, 2, 3.$$

We close the loop by introducing an upper semi circle C_R which runs along $|z| = R$ from $+R$ to $-R$. The singularities enclosed by this loop are

$$z = e^{i\pi/4}, \quad z = e^{3i\pi/4}.$$



By the residue theorem

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{x^4 + 1} dx + \lim_{R \rightarrow \infty} \int_{C_R} \frac{1}{z^4 + 1} dz &= (2\pi i) \left[\text{Res}(f, e^{i\pi/4}) + \text{Res}(f, e^{3i\pi/4}) \right] \\ &= (2\pi i) \left[\frac{1}{4e^{3i\pi/4}} + \frac{1}{4e^{i9\pi/4}} \right] \\ &= \frac{\pi i}{2} \left[e^{-3i\pi/4} + e^{-i\pi/4} \right]. \end{aligned}$$

The C_R term goes to zero, since

$$\left| \int_{C_R} \frac{1}{z^4 + 1} dz \right| \leq \frac{1}{R^4 - 1} \cdot \pi R \rightarrow 0$$

as $R \rightarrow \infty$. Therefore the loop integral becomes

$$\int_{-\infty}^{\infty} \frac{1}{x^4 + 1} dx = \frac{\pi i}{2} \left[e^{-3i\pi/4} + e^{-i\pi/4} \right].$$

We simplify the right-hand side using

$$e^{-3i\pi/4} = \frac{1}{\sqrt{2}}(-1 - i), \quad e^{-i\pi/4} \frac{1}{\sqrt{2}}(1 - i).$$

We obtain

$$\int_{-\infty}^{\infty} \frac{1}{x^4 + 1} dx = \frac{\pi i}{2\sqrt{2}} \left[-2i \right],$$

which simplifies to

$$\int_{-\infty}^{\infty} \frac{1}{x^4 + 1} dx = \frac{\pi}{\sqrt{2}}.$$

10. 6 marks Calculate

$$I = \int_0^\infty \frac{(1 - \cos x)^2}{x^4} dx$$

by complex contour integration.

Solution: We start by invoking some trig identities.

$$(1 - \cos x)^2 = 1 - 2 \cos x + \cos^2 x = 1 - 2 \cos x + \frac{1}{2}(1 + \cos 2x).$$

Therefore

$$I = \frac{1}{2} \int_{-\infty}^\infty \frac{\frac{3}{2} - 2 \cos x + \frac{1}{2} \cos 2x}{x^4} dx$$

since the integrand is even. As usual, we replace the cosine with exponentials for future application of Jordan's lemma.

$$I = \operatorname{Re} \int_{-\infty}^\infty \frac{3 - 4e^{ix} + e^{2ix}}{4x^4} dx.$$

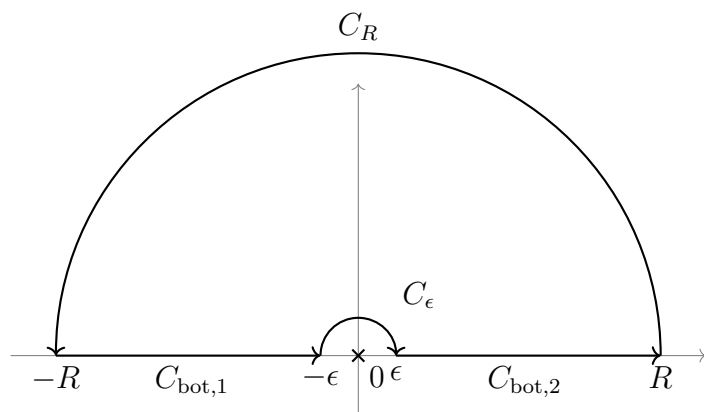
We now complexify the integrand and let

$$f(z) = \frac{3 - 4e^{iz} + e^{2iz}}{4z^4}.$$

Consider the contour

$$C = C_R + C_{\text{bot},1} + C_\epsilon + C_{\text{bot},2},$$

as shown schematically below.



There are no singularities of f enclosed in C . Applying the residue theorem (or just Green's theorem) to the analytic function f inside the contour

C gives the loop equation

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{3 - 4e^{ix} + e^{2ix}}{4x^4} dx + \lim_{R \rightarrow \infty} \int_{C_R} \frac{3 - 4e^{iz} + e^{2iz}}{4z^4} dz \\ & + \lim_{\epsilon \rightarrow 0} \int_{C_\epsilon} \frac{3 - 4e^{iz} + e^{2iz}}{4z^4} dz = 0. \end{aligned} \quad (1)$$

We now consider each piece of the curve C .

- First, note that the integral over C_R tends to zero:

$$\begin{aligned} \left| \int_{C_R} \frac{3 - 4e^{iz} + e^{2iz}}{4z^4} dz \right| & \leq \frac{1}{4R^4} \max_{C_R} \left(3 + 4|e^{iz}| + |e^{2iz}| \right) \text{Length}(C_R) \\ & \leq \frac{1}{4R^4} (3 + 4 + 1)(\pi R) \rightarrow 0 \end{aligned}$$

as $R \rightarrow \infty$. Here we used that $|e^{iz}| = |e^{ix}e^{-y}| = e^{-y} \leq 1$ on the upper-half plane $y \geq 0$.

- Next, compute the integral over C_ϵ . This integral looks difficult to parametrize directly, so we use Laurent expansion.

$$\begin{aligned} \frac{3 - 4e^{iz} + e^{2iz}}{4z^4} & = \frac{1}{4z^4} \left(3 - 4 \left[1 + iz - \frac{z^2}{2} - i\frac{z^3}{3!} + \frac{z^4}{4!} + O(z^5) \right] \right. \\ & \quad \left. + \left[1 + 2iz - 2z^2 - i\frac{4}{3}z^3 + \frac{2}{3}z^4 + O(z^5) \right] \right) \\ & = \frac{1}{4z^4} \left(-i2z - i\frac{2}{3}z^3 + \frac{1}{2}z^4 + O(z^5) \right). \end{aligned}$$

Therefore

$$\int_{C_\epsilon} \frac{3 - 4e^{iz} + e^{2iz}}{4z^4} dz = \int_{C_\epsilon} \left(-\frac{i}{2}z^{-3} - \frac{i}{6}z^{-1} + \frac{1}{8} + O(z) \right) dz.$$

We now parametrize C_ϵ by setting $z(t) = \epsilon e^{it}$ with $t : \pi \rightarrow 0$.

$$\int_{C_\epsilon} \frac{3 - 4e^{iz} + e^{2iz}}{4z^4} dz = \int_{\pi}^0 \left(-\frac{i}{2\epsilon^3}e^{-3it} - \frac{i}{6\epsilon}e^{-it} + \frac{1}{8} + O(\epsilon) \right) (i\epsilon e^{it}) dt.$$

We note that

$$\int_{\pi}^0 \left(-\frac{i}{2\epsilon^3}e^{-3it} \right) (i\epsilon e^{it}) dt = 0.$$

We also note that upon taking $\epsilon \rightarrow 0$, many terms vanish.

$$\lim_{\epsilon \rightarrow 0} \int_{C_\epsilon} \frac{3 - 4e^{iz} + e^{2iz}}{4z^4} dz = \int_{\pi}^0 \left(-\frac{i}{6}e^{-it} \right) (i\epsilon e^{it}) dt = -\frac{\pi}{6}.$$

This completes the calculation of the contribution from the small semi-circle C_ϵ .

• We can now substitute the contributions of C_R and C_ϵ into (1). The result is

$$\int_{-\infty}^{\infty} \frac{3 - 4e^{ix} + e^{2ix}}{4x^4} dx + 0 - \frac{\pi}{6} = 0.$$

Therefore

$$I = \frac{\pi}{6}.$$

11. 6 marks Let $n \geq 2$ be an integer and α be such that $0 < \alpha < n$. Calculate

$$\int_0^\infty \frac{x^{\alpha-1}}{x^n + 1} dx$$

by complex contour integration.

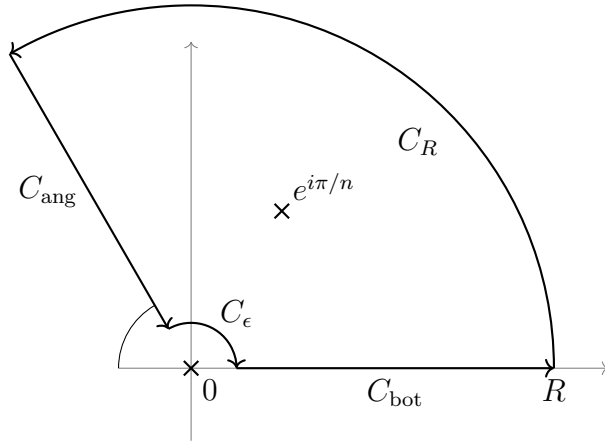
Solution: Let

$$f(z) = \frac{P}{Q} = \frac{z^{\alpha-1}}{z^n + 1}$$

where $z^{\alpha-1}$ denotes the principal branch. Note that $f(z)$ is possibly undefined at $z = 0$, and that $f(z)$ has singularities at $e^{\pi i/n} \omega_n^k$ where ω_n is the n -th root of unity. Consider the closed loop

$$C = C_R + C_{\text{ang}} + C_\epsilon + C_{\text{bot}}$$

as shown schematically below. Here C_{ang} is a straight line coming into the origin along the ray $z(t) = te^{2\pi i/n}$ with t decreasing from infinity.



There is a single pole enclosed, so by the residue theorem

$$\begin{aligned} \int_C \frac{z^{\alpha-1}}{z^n + 1} dz &= (2\pi i) \text{Res}(f, e^{i\pi/n}) \\ &= (2\pi i) \frac{P(e^{i\pi/n})}{Q'(e^{i\pi/n})} \\ &= (2\pi i) \frac{e^{i(\alpha-1)\pi/n}}{ne^{i(n-1)\pi/n}} \\ &= \frac{2\pi i}{n} e^{i\alpha\pi/n} e^{-i\pi} \\ &= -\frac{2\pi i}{n} e^{i\alpha\pi/n}. \end{aligned}$$

Note that the integral over C_R tends to zero:

$$\left| \int_{C_R} \frac{z^{\alpha-1}}{z^n + 1} dz \right| \leq \frac{R^{\alpha-1}}{R^n - 1} \cdot \pi R \rightarrow 0$$

as $R \rightarrow \infty$. Second, note that the integral over C_ϵ tends to zero:

$$\left| \int_{C_\epsilon} \frac{z^{\alpha-1}}{z^n + 1} dz \right| \leq \frac{\epsilon^{\alpha-1}}{1 - \epsilon^n} \cdot \pi \epsilon \rightarrow 0$$

as $\epsilon \rightarrow 0$. Next, as $R \rightarrow \infty$ and $\epsilon \rightarrow 0$, the integral over C_{ang} becomes

$$\int_0^0 \frac{z(t)^{\alpha-1}}{z(t)^n + 1} dz(t), \quad z(t) = te^{2\pi i/n}.$$

Therefore

$$\int_{C_{\text{ang}}} \frac{z^{\alpha-1}}{z^n + 1} dz \rightarrow - \int_0^\infty \frac{t^{\alpha-1} e^{2\pi(\alpha-1)i/n}}{t^n + 1} e^{2\pi i/n} dt.$$

Using this, the loop integral residue equation becomes

$$\int_0^\infty \frac{x^{\alpha-1}}{x^n + 1} dx - e^{2\pi i/n} e^{2\pi i(\alpha-1)/n} \int_0^\infty \frac{x^{\alpha-1}}{x^n + 1} dx = -\frac{2\pi i}{n} e^{i\alpha\pi/n}.$$

We multiply this through by $e^{-i\alpha\pi/n}$ and simplify.

$$(e^{-i\alpha\pi/n} - e^{i\alpha\pi/n}) \int_0^\infty \frac{x^{\alpha-1}}{x^n + 1} dx = -\frac{2\pi i}{n}.$$

This brings out sine.

$$(-2i)(\sin(\alpha\pi/n)) \int_0^\infty \frac{x^{\alpha-1}}{x^n + 1} dx = -\frac{2\pi i}{n}.$$

Therefore

$$\int_0^\infty \frac{x^{\alpha-1}}{x^n + 1} dx = \frac{\pi}{n \sin(\alpha\pi/n)}.$$