

Mathematics 305 — Final Exam

Practice Final D

- The test consists of 12 pages and 11 questions worth a total of 60 marks.
- Electronic devices of any kind (including calculators, cell phones, etc.) are forbidden.
- No work on this page will be marked.
- If you require extra paper then put up your hand and ask your instructor. You must indicate your name, student number, test number and question number on any extra pages.

Student number								
Section								
Name								
Signature								

1. (a) 4 marks Simplify

$$\cos[i \log(-1 + i)]$$

into the form $a + ib$.

(b) 4 marks Simplify

$$\text{Log}\left(\frac{\sqrt{3} + i}{\sqrt{2}(1 + i)}\right)$$

into the form $a + ib$.

2. 6 marks Find a branch of $(z - 1)^{-1/2}$ that is analytic on $\mathbb{C} \setminus S$, where

$$S = \{z : \operatorname{Re}(z) = 1, \operatorname{Im}(z) \leq 0\}.$$

Using this branch, compute

$$\int_C (z - 1)^{-1/2} dz$$

where C is the semicircle centered at 1 going counterclockwise from 2 to 0.

3. 5 marks Express the residue at 0 of

$$\frac{e^{1/z}}{1-z}$$

as an infinite sum, and calculate this sum.

4. 4 marks Compute

$$\int_C |z|^2 dz$$

where C is the directed line segment from i to $1 + i$.

5. 4 marks Compute

$$\int_C \frac{100! e^{iz}}{(z+1)^{100}} dz$$

where $C = \{|z| = 5\}$ is a counterclockwise circle.

6. 4 marks Compute

$$\int_C \frac{z \cos z}{(2z - \pi)^2} dz$$

where $C = \{|z| = 4\}$ is a counterclockwise circle.

7. 6 marks Find the Laurent series of

$$f(z) = \frac{z}{z^2 - z - 2}$$

centered at $z_0 = 0$ that converges for $|z| > 2$. Explicitly calculate the first two terms, then compute $\int_{|z|=4} f(z) dz$ where the circle $|z| = 4$ is oriented counterclockwise.

8. 5 marks Find the number of zeros, counted with multiplicity, of

$$f(z) = z^5 + 3z + 1$$

in the disk $\{|z| < 2\}$.

9. 6 marks Calculate

$$\int_{-\infty}^{\infty} \frac{1}{x^4 + 1} dx$$

by complex contour integration.

10. 6 marks Calculate

$$I = \int_0^{\infty} \frac{(1 - \cos x)^2}{x^4} dx$$

by complex contour integration.

11. 6 marks Let $n \geq 2$ be an integer and α be such that $0 < \alpha < n$. Calculate

$$\int_0^\infty \frac{x^{\alpha-1}}{x^n + 1} dx$$

by complex contour integration.