

Mathematics 305 — Midterm 2

Practice Midterm B

- The test consists of 6 pages and 4 questions worth a total of 40 marks.
- Electronic devices of any kind (including calculators, cell phones, etc.) are forbidden.
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

Student number								
Section								
Name								
Signature								

Please do not write on this page — it will not be marked.

Additional instructions

- Please use the spaces indicated.
- If you require extra paper then put up your hand and ask your instructor.
 - You must put your name and student number on any extra pages.
 - You must indicate the test-number and question-number.
 - Please do this **on both sides** of any extra pages.
- Please do not dismember your test. You must submit all pages.
- Smoking is strictly prohibited during the test.

1. 10 marks

(a) Compute

$$\int_C \bar{z}^2 \cos z \, dz$$

where C is the circle $|z| = \pi$ running counterclockwise.

Solution: We notice that along C , we have $z\bar{z} = \pi^2$ and so $\bar{z} = \pi^2/z$. The integral is therefore

$$\int_C \frac{\pi^4}{z^2} \cos(z) \, dz.$$

We compute the residue at the origin by Laurent expansion.

$$\frac{\pi^4}{z^2} \cos(z) = \frac{\pi^4}{z^2} \left[1 - \frac{z^2}{2} + \frac{z^4}{4!} - \cdots \right].$$

There is no coefficient in front of $1/z$, hence

$$\int_C \frac{\pi^4}{z^2} \cos(z) \, dz = 2\pi i \operatorname{Res}\left(\frac{\pi^4}{z^2} \cos(z), 0\right) = 0.$$

(b) Let $f(z)$ be analytic on $|z| \leq 2$ and suppose

$$\max_{|z|=2} |f| \leq 1.$$

Suppose $f(z)$ has a zero of order 3 at $z = 0$. Find the largest possible of $|f(1)|$. Hint: consider $g(z) = f(z)/z^3$.

Solution: The function $g(z)$ is analytic on $|z| \leq 2$, and

$$\max_{|z|=2} |g| = \max_{|z|=2} \frac{|f|}{|z|^3} \leq \frac{1}{8}.$$

Therefore $|g(z)| \leq 1/8$ on all of $|z| \leq 2$ by the maximum principle. It follows that

$$\frac{|f(1)|}{1^3} = |g(1)| \leq 1/8$$

and so $|f(1)| \leq 1/8$ for any function $f(z)$ as specified in the question. The bound is saturated by the particular function $f(z) = \frac{1}{8}z^3$.

2. 10 marks

(a) Compute

$$\int_C \frac{1}{z^2(z^4 + 2)} dz$$

where $C = \{|z| = 2\}$ oriented counterclockwise.

Solution: There are no zeros outside the circle $|z| = 2$. By Green's theorem (using that the 2D curl is $2i\bar{\partial}_z f = 0$), we may deform the circle $|z| = 2$ to the large circle $C_R = \{|z| = R\}$ for $R \gg 1$.

$$\int_C \frac{1}{z^2(z^4 + 2)} dz = \int_{C_R} \frac{1}{z^2(z^4 + 2)} dz$$

We can estimate the integral by

$$\left| \int_{C_R} \frac{1}{z^2(z^4 + 2)} dz \right| \leq \left(\max_{|z|=R} \frac{1}{|z|^2|z^4 + 2|} \right) \text{Length}(C_R).$$

Using the reverse triangle inequality $|z^4 + 2| \geq R^4 - 2$ for $R \gg 1$ gives

$$\left| \int_{C_R} \frac{1}{z^2(z^4 + 2)} dz \right| \leq \left(\frac{1}{R^2(R^4 - 2)} \right) (2\pi R) \rightarrow 0$$

as $R \rightarrow \infty$. Hence

$$\int_C \frac{1}{z^2(z^4 + 2)} dz = 0.$$

(b) Suppose the Taylor series of $f(z) = \frac{1}{1 + e^z}$ at the point $z_0 = i$ is given by the infinite series

$$\sum_{n=0}^{\infty} a_n (z - i)^n.$$

What is the radius of convergence of this series?

Solution: The radius of convergence is the distance from the reference point $z_0 = i$ to the nearest singularity. The singularities of the meromorphic function $f(z)$ occur when

$$1 + e^z = 0 \implies e^z = -1,$$

which occurs at $z = \pm i\pi, \pm i(3\pi), \pm i(5\pi), \dots$. The nearest singularity to i is at $i\pi$. Therefore the series converges on

$$|z - i| < \pi - 1$$

and the radius of convergence is $\pi - 1$.

3. 10 marks Let $a > 1$. Compute

$$\int_0^{2\pi} \frac{1}{a + \cos \varphi} d\varphi$$

by transforming the integral into a complex contour integral.

Solution: We let $z = e^{i\varphi}$, so that $dz = iz d\varphi$ and $\cos \varphi = \frac{1}{2}(z + z^{-1})$. The integral becomes

$$\int_{|z|=1} \frac{1}{a + \frac{1}{2}(z + z^{-1})} \frac{dz}{iz}.$$

This simplifies to

$$\int_{|z|=1} \frac{-2i}{z^2 + 2az + 1} dz.$$

To apply the residue theorem, we must find the roots of

$$z^2 + 2az + 1 = 0 \implies z = \frac{1}{2} \left[-2a \pm \sqrt{4a^2 - 4} \right].$$

These two roots are

$$z_+ = -a + \sqrt{a^2 - 1}, \quad z_- = -a - \sqrt{a^2 - 1}.$$

The product of these is equal to one: $z_+ z_- = 1$. This can be seen by direct calculation, or noticing that

$$z^2 + 2az + 1 = 0 \implies 1 + 2az^{-1} + (z^{-1})^2 = 0$$

by multiplying the polynomial by z^{-2} . This means that if z is a root, then $1/z$ is also a root, and their product is one. With this observation, since

$$|z_-| = a + \sqrt{a^2 - 1} > 1$$

we have that $|z_+| < 1$ since $|z_+| = |z_-|^{-1}$. Therefore

$$\int_{|z|=1} \frac{-2i}{z^2 + 2az + 1} dz = (2\pi i) \operatorname{Res} \left(\frac{P}{Q}, z_+ \right), \quad \frac{P}{Q} = \frac{-2i}{z^2 + 2az + 1}.$$

We have

$$\operatorname{Res} \left(\frac{P}{Q}, z_+ \right) = \frac{-2i}{Q'(z_+)} = \frac{-2i}{2z_+ + 2a} = \frac{-i}{\sqrt{a^2 - 1}}.$$

Therefore

$$\int_{|z|=1} \frac{-2i}{z^2 + 2az + 1} dz = \frac{2\pi}{\sqrt{a^2 - 1}}$$

which equals the integral requested in the question.

4. 10 marks Let

$$f(z) = \frac{z+1}{z^2-3z+2}.$$

Consider the Laurent series of $f(z)$ centered at $z_0 = 0$ that converges for $|z| > 2$. Evaluate the first two terms and compute

$$\int_{|z|=4} f(z) dz$$

where $|z| = 4$ runs counterclockwise.

Solution: We factor the function and search for its partial fractions decomposition:

$$f = \frac{z+1}{(z-2)(z-1)} = \frac{A}{z-2} + \frac{B}{z-1}.$$

The coefficients A, B are found by multiplying both sides by $(z-2)(z-1)$, which leads to

$$z+1 = A(z-1) + B(z-2).$$

Substituting $z = 1$ gives $B = -2$ and substituting $z = 2$ gives $A = 3$. Therefore

$$f(z) = \frac{3}{z-2} - \frac{2}{z-1} = \frac{3}{z} \frac{1}{1-2z^{-1}} - \frac{2}{z} \frac{1}{1-z^{-1}}.$$

By the formula for a geometric series, we obtain

$$f(z) = \frac{3}{z} \left[\sum_{n=0}^{\infty} (2/z)^n \right] - \frac{2}{z} \left[\sum_{n=0}^{\infty} \frac{1}{z^n} \right].$$

The first two terms are

$$f(z) = \frac{3}{z} \left[1 + \frac{2}{z} \right] - \frac{2}{z} \left[1 + \frac{1}{z} \right] + \dots$$

which becomes

$$f(z) = \frac{1}{z} + \frac{4}{z^2} + \dots$$

This Laurent expansion holds for $|z| > 2$, and hence holds along the circle $|z| = 4$. Therefore to integrate $f(z)$, we can integrate each term in the series. Since

$$\int_{|z|=4} z^m dz = \begin{cases} 0 & \text{if } m \neq -1, \\ 2\pi i & \text{if } m = -1, \end{cases}$$

we conclude

$$\int_{|z|=4} f(z) dz = 2\pi i.$$