

# Mathematics 305 — Midterm 2

## Practice Midterm D

- The test consists of 8 pages and 4 questions worth a total of 48 marks.
- Electronic devices of any kind (including calculators, cell phones, etc.) are forbidden.
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

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| Student number |       |  |  |  |  |  |  |  |
| Section        |       |  |  |  |  |  |  |  |
| Name           | ..... |  |  |  |  |  |  |  |
| Signature      |       |  |  |  |  |  |  |  |

*Please do not write on this page — it will not be marked.*

## **Additional instructions**

- Please use the spaces indicated.
- If you require extra paper then put up your hand and ask your instructor.
  - You must put your name and student number on any extra pages.
  - You must indicate the test-number and question-number.
  - Please do this **on both sides** of any extra pages.
- Please do not dismember your test. You must submit all pages.
- Smoking is strictly prohibited during the test.

1. 10 marks Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be an analytic function. Suppose  $\operatorname{Im} f(z) \geq 0$  for all  $z \in \mathbb{C}$ . Prove that  $f$  is a constant function. To do this, consider

$$g(z) = e^{if(z)}$$

and apply Liouville's theorem.

**Solution:** Write  $f(z) = u + iv$  as usual. We consider  $g(z)$  as prompted and notice

$$|g(z)| = |e^{iu}e^{-v}| = e^{-v}.$$

Since  $\operatorname{Im} f(z) \geq 0$ , this means  $v \geq 0$  and so  $e^{-v} \leq 1$ . Therefore

$$|g(z)| \leq 1.$$

Since  $g(z)$  is an analytic function on all of  $\mathbb{C}$  which is bounded, by Liouville's theorem we know that  $g(z)$  is a constant function. Therefore

$$e^{if(z)} = c$$

for some non-zero constant  $c \in \mathbb{C}$ . Write  $c = r_0 e^{i\varphi_0}$ . Recall that  $e^{z_1} = e^{z_2}$  if and only if  $z_1 = z_2 + (2\pi i)k$  for some integer  $k$ . Then we have proved that

$$if(z) = \ln r_0 + i\varphi_0 + (2\pi i)k(z), \quad k(z) = 0, \pm 1, \pm 2, \dots$$

Since  $f(z)$  is a continuous function, the integer  $k$  cannot jump around and must be constant. Therefore  $f(z)$  is constant.

2. 9 marks (a) 3 marks Compute

$$\int_C \frac{e^z}{\sin(z)(z-1)^3} dz$$

where  $C$  is the circle  $|z+2|=1$  running counterclockwise.

**Solution:** The denominator is zero when

$$z = 1 \quad \text{or} \quad z = \pi k, \quad k = 0, \pm 1, \pm 2, \dots$$

The inside of the circle  $C$  is described by

$$|z+2| \leq 1$$

and none of these singular points are contained inside the circle  $C$ . Therefore the integrand is an analytic function inside  $C$  and so

$$\int_C \frac{e^z}{\sin(z)(z-1)^3} dz = 0.$$

- (b) 3 marks Compute

$$\int_C \frac{1}{3i + \sqrt{z}} dz$$

where  $C$  is the circle  $|z-2|=1$  running counterclockwise.

**Solution:** The principal branch  $\sqrt{z}$  is analytic on  $\mathbb{C} \setminus (-\infty, 0]$ , and so  $\sqrt{z}$  is analytic inside  $C$ . The denominator of  $f(z)$  is zero when

$$\sqrt{z} = -3i \implies z = 9i^2 = -9.$$

At this singular point  $z = -9$ , we have

$$|z-2| = 11 > 1.$$

Therefore the integrand is analytic inside  $C$  and so

$$\int_C \frac{1}{3i + \sqrt{z}} dz = 0.$$

- (c) 3 marks Compute

$$\int_C \frac{1}{(z-3)(z-5)} dz$$

where  $C$  is the circle  $|z-2|=2$  running counterclockwise.

**Solution:** The denominator is zero at  $z = 3$  and  $z = 5$ , but only

$$z = 3$$

is inside the curve  $C$ , namely satisfies  $|z-2| < 2$ . Write the integrand as

$$\frac{g(z)}{(z-3)}, \quad g(z) = \frac{1}{(z-5)}.$$

By the Cauchy integral formula,

$$\int_C \frac{1}{(z-3)(z-5)} dz = 2\pi i \cdot g(3),$$

and so

$$\int_C \frac{1}{(z-3)(z-5)} dz = -\pi i.$$

3. 10 marks Calculate

$$I = \int_0^{\infty} \frac{\sin x}{x(x^2 + 1)} dx$$

by complex contour integration.

**Solution:** The integrand is even, and so the integral is equal to

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2 + 1)} dx.$$

Consider

$$f(z) = \frac{P}{Q} = \frac{e^{iz}}{z(z^2 + 1)}.$$

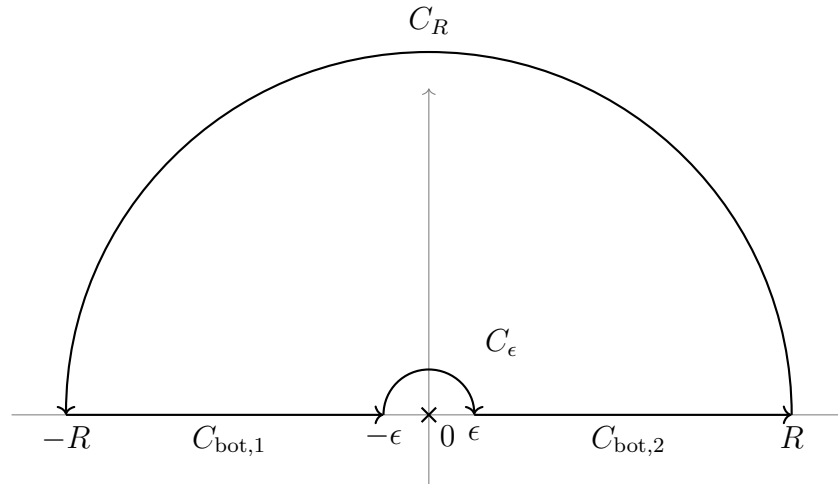
Here we replaced the sine with the exponential for later application of Jordan's lemma. The singularities of  $f(z)$  are at

$$z = 0, \quad z = \pm i.$$

Consider the contour

$$C = C_R + C_{\text{bot},1} + C_{\epsilon} + C_{\text{bot},2},$$

as shown schematically below.



The only singularity of  $f$  enclosed in  $C$  is  $+i$ . By the residue theorem

$$\begin{aligned} \int_C \frac{e^{iz}}{z(z^2 + 1)} dz &= (2\pi i) \text{Res}(f, i) \\ &= (2\pi i) \frac{P(i)}{Q'(i)} \\ &= (2\pi i) \frac{e^{-1}}{i(2i)} \\ &= -i(\pi e^{-1}). \end{aligned}$$

We now consider each piece of the curve  $C$ . First, note that the integral over  $C_R$  tends to zero:

$$\begin{aligned} \left| \int_{C_R} \frac{e^{iz}}{z(z^2+1)} dz \right| &\leq \left( \frac{1}{R(R^2-1)} \max_{C_R} |e^{iz}| \right) \text{Length}(C_R) \\ &\leq \frac{1}{R(R^2+1)} (\pi R) \rightarrow 0 \end{aligned}$$

as  $R \rightarrow \infty$ . Here we used that  $|e^{iz}| = |e^{ix}e^{-y}| = e^{-y} \leq 1$  on the upper-half plane  $y \geq 0$ . Next, compute the integral over  $C_\epsilon$  by setting  $z(t) = \epsilon e^{it}$  with  $t$  ranging from  $\pi$  to zero:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{C_\epsilon} \frac{e^{iz}}{z(z^2+1)} dz &= \lim_{\epsilon \rightarrow 0} \int_{\pi}^0 \frac{e^{i\epsilon \exp(it)}}{\epsilon e^{it}(\epsilon^2 e^{2it} + 1)} (\epsilon i) e^{it} dt \\ &= - \int_0^{\pi} i dt \\ &= -i\pi. \end{aligned}$$

The contour integral

$$\int_{C_R + C_{\text{bot}} + C_\epsilon} \frac{e^{iz}}{z(z^2+1)} dz = (2\pi i) \text{Res}(f, i),$$

upon taking  $R \rightarrow \infty$  and  $\epsilon \rightarrow 0$ , becomes

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)} dx - i\pi = -i\pi e^{-1}.$$

Therefore

$$\text{Im p.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)} dx = \pi(1 - e^{-1}).$$

Since  $\text{Im } e^{ix} = \sin x$ , we conclude

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)} dx = \frac{\pi}{2}(1 - e^{-1}).$$

4. 10 marks Find the Laurent series for

$$f(z) = \frac{z^2}{z^2 - z - 2}$$

centered at 0 which converges at  $z = \frac{3i}{2}$ . Write your answer as

$$a_0 + \sum_{k=1}^{\infty} a_k z^k + \sum_{k=1}^{\infty} a_{-k} z^{-k}.$$

What is  $a_0$  and  $a_{-1}$ ?

**Solution:** We write

$$f(z) = \frac{z^2 - z - 2 + z + 2}{z^2 - z - 2}.$$

We factor the denominator and search for partial fractions decomposition.

$$f(z) = 1 + \frac{z + 2}{(z - 2)(z + 1)} = 1 + \frac{A}{z - 2} + \frac{B}{z + 1}.$$

Therefore

$$z + 2 = A(z + 1) + B(z - 2),$$

which implies

$$4 = 3A \implies A = \frac{4}{3}, \quad 1 = -3B \implies B = -\frac{1}{3}.$$

Therefore

$$f(z) = 1 + \frac{4}{3} \frac{1}{z - 2} - \frac{1}{3} \frac{1}{z + 1}.$$

The given point  $\frac{3i}{2}$  is in the region  $1 < |z| < 2$ . We manipulate the expression to prepare for geometric series expansion.

$$f(z) = 1 - \frac{4}{3 \cdot 2} \frac{1}{1 - \frac{z}{2}} - \frac{1}{3z} \frac{1}{1 + z^{-1}}.$$

Since  $|z/2| < 1$  and  $|z^{-1}| < 1$  in the region under consideration, we may use the geometric series expansion.

$$f(z) = 1 - \frac{4}{3 \cdot 2} \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^k - \frac{1}{3z} \sum_{k=0}^{\infty} (-1)^k z^{-k}.$$

This simplifies to

$$f(z) = \frac{1}{3} + \sum_{k=1}^{\infty} \left[ \frac{-1}{3 \cdot 2^{k-1}} \right] z^k + \sum_{k=1}^{\infty} \left[ \frac{(-1)^k}{3} \right] z^{-k}.$$

Therefore  $a_0 = 1/3$  and  $a_{-1} = -1/3$ .