1. Smooth Manifolds

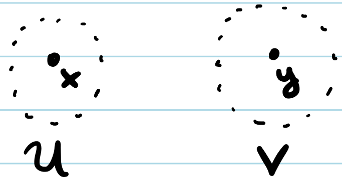
Def: A topological manifold of dimension n is a paracompact, Hausdorff space s.t. each pt has a nbhd homeomorphic to an open set in $\mathbb{R}^n$.

Paracompact: Every open cover has a locally finite refinement.

Refinement: A cover by open sets which are subsets of the original cover.

Locally finite: Every pt has a nbhd that intersects only finitely many sets.

Hausdorff: $\forall x \neq y$, $\exists\, \mathcal{U}, V$ open with: $x \in \mathcal{U}$, $y \in V$, $\mathcal{U} \cap V = \phi$



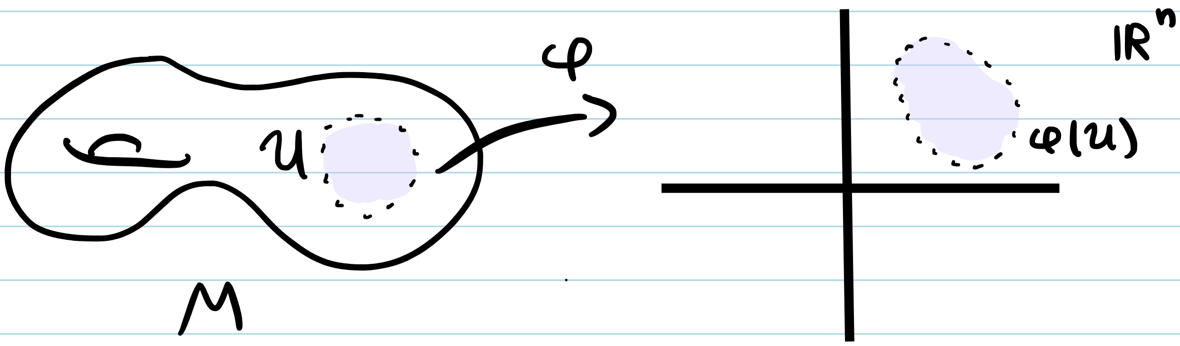
Homeomorphism: $f: X \to Y$ bijection with f and f^{-1} continuous.

This course is about smooth manifolds. Need a bit more terminology.

Def: M topological mfd.

A coordinate chart on M is a pair $(\mathcal{U}, \varphi)$ where:

- $\mathcal{U} \subseteq M$ open
- $\varphi: \mathcal{U} \to \varphi(\mathcal{U}) \subseteq \mathbb{R}^n$ homeomorphism

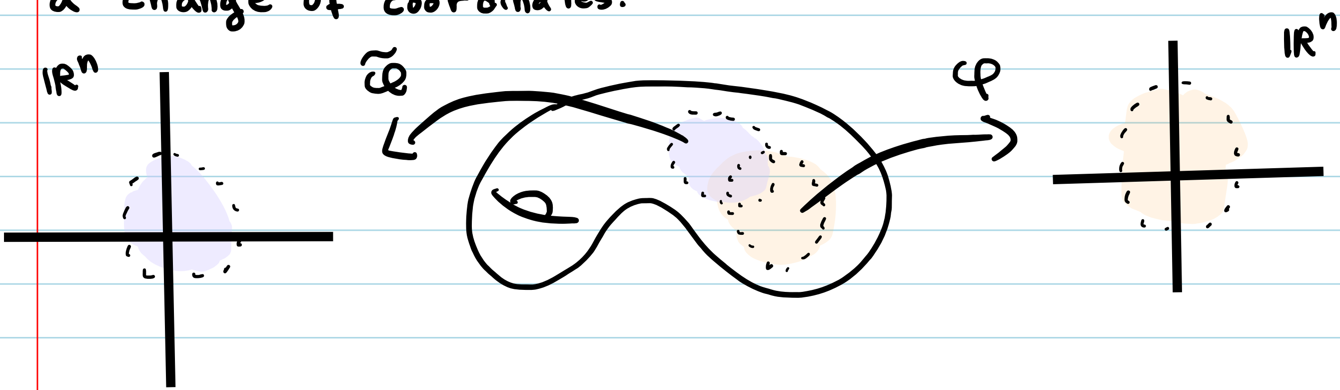


Def: An atlas for M is an open cover

$$M = \bigcup_{\alpha \in \mathcal{A}} \mathcal{U}_\alpha \quad \text{s.t. each } \mathcal{U}_\alpha \text{ is associated to a coord chart } (\mathcal{U}_\alpha, \varphi_\alpha).$$

Def: Let $\mathcal{A}$ be an atlas for M
Let $(\mathcal{U}, \varphi), (\tilde{\mathcal{U}}, \tilde{\varphi}) \in \mathcal{A}$

The map $\tilde{\varphi} \circ \varphi^{-1} : \varphi(\mathcal{U} \cap \tilde{\mathcal{U}}) \to \tilde{\varphi}(\mathcal{U} \cap \tilde{\mathcal{U}})$ is called a change of coordinates.



Def: A smooth manifold is a topological manifold with a smooth structure. defined below

Smooth structure: an equivalence class of atlases s.t. all change of coordinates are smooth with smooth inverse.

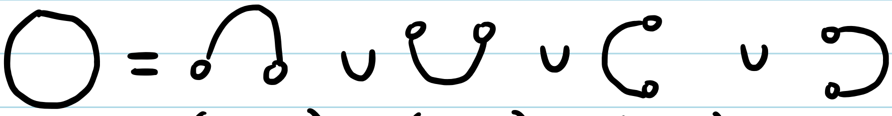
The equivalence relation is:
$\mathcal{A} \sim \mathcal{A}'$ if $\forall\, (\mathcal{U}, \varphi) \in \mathcal{A}$
$(V, \psi) \in \mathcal{A}'$
then $\varphi \circ \psi^{-1} : \psi(\mathcal{U} \cap V) \to \varphi(\mathcal{U} \cap V)$
is smooth with smooth inverse.
(only consider pairs with common domain)

Smooth function: A map $f : \underset{\subseteq \mathbb{R}^n}{\mathcal{U}} \to \underset{\subseteq \mathbb{R}^k}{V}$ is smooth
if partial derivatives of all orders exist and are continuous.

With this jargon out of the way, we move on to learning by examples.

ex) $S^1 = \{x^2+y^2=1\} \subseteq \mathbb{R}^2$

Atlas:



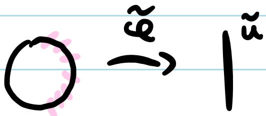
$\{y>0\}$ $\{y<0\}$ $\{x<0\}$ $\{x>0\}$

$\mathcal{U} = \{y>0\} \cap S^1$

$\varphi(u, \sqrt{1-u^2}) = u$

$\tilde{\mathcal{U}} = \{x>0\} \cap S^1$

$\tilde{\varphi}(\sqrt{1-\tilde{u}^2}, \tilde{u}) = \tilde{u}$



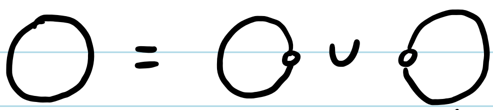
Change of coords:

$\tilde{\varphi} \circ \varphi^{-1}(u) = \sqrt{1-u^2}$ smooth on $\mathcal{U} \cap \tilde{\mathcal{U}}$.

usually just write: $\tilde{u} = \sqrt{1-u^2}$

Omitted: can also compute change of coords between all other charts.

Another atlas:



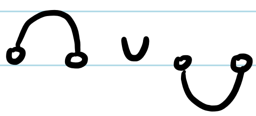
$\mathcal{U}$ $\tilde{\mathcal{U}}$

$\mathcal{U} = \{e^{i\theta} : 0 < \theta < 2\pi\}$

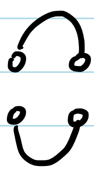
$\varphi(e^{i\theta}) = \theta$

$\tilde{\mathcal{U}} = \{e^{i\tilde{\theta}} : -\pi < \tilde{\theta} < \pi\}$

$\tilde{\varphi}(e^{i\tilde{\theta}}) = \tilde{\theta}$

change of coords: $\mathcal{U} \cap \tilde{\mathcal{U}} =$ 

$\tilde{\varphi} \circ \varphi^{-1}(\theta) = \begin{cases} \theta & \text{on} \\ \theta - 2\pi & \text{on} \end{cases}$



just write: $\tilde{\theta} = \begin{cases} \theta & \text{on} \\ \theta - 2\pi & \text{on} \end{cases}$  .

Exercise: These two atlases give the same smooth structure.

Exercise: Cannot cover S^1 by a single coord chart.

More notation:

- Instead of $(\mathcal{U}, \varphi)$, often write coord chart as: $(\mathcal{U}, x^i)$

$$\varphi(p) = (x^1, \dots, x^n) \in \mathbb{R}^n$$

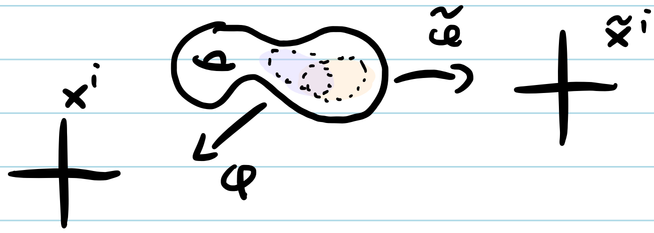
convention: label coords with upper indices x^i instead of x_i.

- $(\mathcal{U}, \varphi)$, $(\tilde{\mathcal{U}}, \tilde{\varphi})$ overlapping charts.

Write change of coords as:

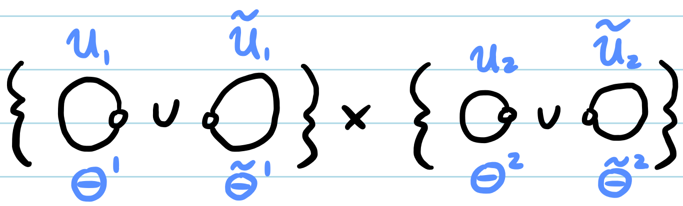
$$\tilde{x}^i = f^i(x^1, \dots, x^n), \qquad f = \tilde{\varphi} \circ \varphi^{-1}$$

$$f : \varphi(\mathcal{U} \cap \tilde{\mathcal{U}}) \to \tilde{\varphi}(\mathcal{U} \cap \tilde{\mathcal{U}})$$



ex) $T^2 = S^1 \times S^1$

Atlas =



e.g. chart: $\{\mathcal{U}_1 \times \mathcal{U}_2, (\theta^1, \theta^2)\}$

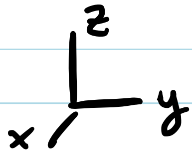
$\{\tilde{\mathcal{U}}_1 \times \tilde{\mathcal{U}}_2, (\tilde{\theta}^1, \tilde{\theta}^2)\}$

change of coords:

$$\tilde{\theta}^1 = \begin{cases} \theta^1 \\ \theta^1 - 2\pi \end{cases} \qquad \tilde{\theta}^2 = \begin{cases} \theta^2 \\ \theta^2 - 2\pi \end{cases}$$

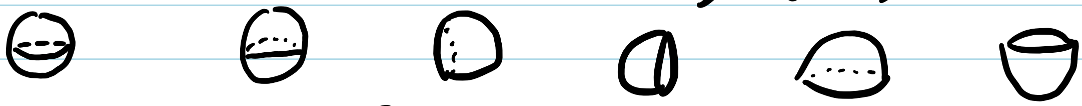
More generally: Given mfd M, N, can define $M \times N$ by generalizing this construction.

ex) $S^2 = \{x^2+y^2+z^2=1\} \subseteq \mathbb{R}^3$



Charts:

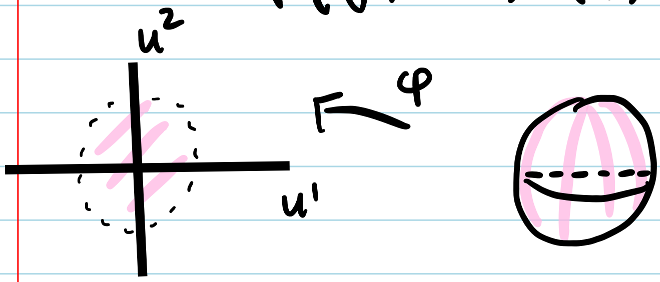
 $= \{x>0\} \cup \{x<0\} \cup \{y>0\} \cup \{y<0\} \cup \{z>0\} \cup \{z<0\}$



e.g. let $\mathcal{U} = \{x>0\} \cap S^2$

$$\mathcal{U} = \{(\sqrt{1-(u^1)^2-(u^2)^2}, u^1, u^2) : (u^1, u^2) \in B_1(0)\}$$

$$\varphi(\sqrt{1-(u^1)^2-(u^2)^2}, u^1, u^2) = (u^1, u^2)$$



front half

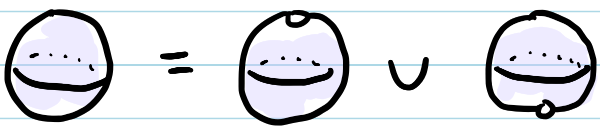
let $\tilde{\mathcal{U}} = \{y<0\} \cap S^2$

$$\tilde{\varphi}(\tilde{u}^1, -\sqrt{1-(\tilde{u}^1)^2-(\tilde{u}^2)^2}, \tilde{u}^2) = (\tilde{u}^1, \tilde{u}^2)$$

change of coords : $\begin{cases} \tilde{u}^1 = \sqrt{1-(u^1)^2-(u^2)^2} \\ \tilde{u}^2 = u^2 \end{cases}$

Other coord charts are similar.

Another common set of charts: "stereographic projection"



$S^2 \quad = \quad S^2 \setminus \{N\} \quad \cup \quad S^2 \setminus \{S\}$

will be on homework 1.

ex) $GL(n,\mathbb{R}) = \{ n\times n \text{ matrices with } \det \neq 0 \}$

$\det : \underset{\cong \mathbb{R}^{n^2}}{Mat(n\times n)} \to \mathbb{R}$ is continuous

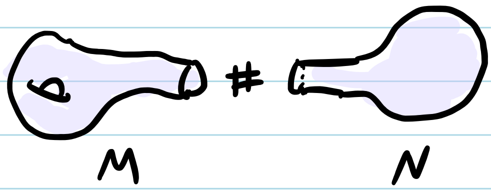
$\Rightarrow \det^{-1}(0) \subseteq \mathbb{R}^{n^2}$ is closed

$\Rightarrow GL(n,\mathbb{R})$ is an open set in $\mathbb{R}^{n^2}$

$\Rightarrow GL(n,\mathbb{R})$ is a mfd of dim n^2 covered by a single chart.

ex) Projective space $\mathbb{R}\mathbb{P}^n$. <u>Omitted</u>: will be on homework 1.

ex) Connected sums: let M, N be mfd



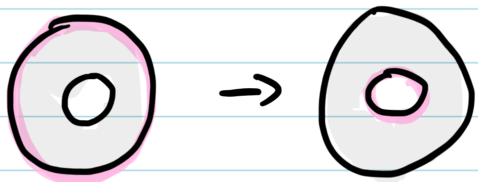
$M \# N$

$p \in M$ in chart $(\mathcal{U}, \varphi)$ with $\varphi(p) = 0$
$q \in N$ in chart (V, ψ) with $\psi(p) = 0$

$B_{2\varepsilon}(0) \subseteq \varphi(\mathcal{U}), \quad B_{2\varepsilon}(0) \subseteq \psi(V)$

$$g : \{ \varepsilon < |x| < 2\varepsilon \} \to \{ \varepsilon < |x| < 2\varepsilon \}$$
$$x \mapsto \frac{2\varepsilon^2}{|x|^2} x$$

note: $\{ |x| = 2\varepsilon \} \leftrightarrow \{ |x| = \varepsilon \}$



$$M \# N = \Big(M \setminus \overset{\varphi(\mathcal{U})\ \supseteq}{\{|x| \le \varepsilon\}} \sqcup N \setminus \overset{\psi(V)\ \supseteq}{\{|x| \le \varepsilon\}} \Big) \Big/ \sim$$

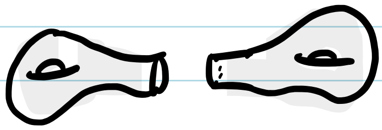
with glueing $\underset{\in\,\varphi(\mathcal{U})}{x} \sim \underset{\in\,\psi(V)}{g(x)}$.

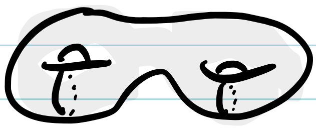
In terms of change of coords: $(\underset{\subseteq M}{\mathcal{U}}, x^i)$, $(\underset{\subseteq N}{V}, \tilde{x}^i)$

Declare: $\tilde{x}^i = \dfrac{2\varepsilon^2}{|x|^2} x^i$

on region $\{\varepsilon < |x| < 2\varepsilon\} = \{\varepsilon < |\tilde{x}| < 2\varepsilon\}$.

ex) Genus 2 surface $= T^2 \# T^2$

$=$ 

$=:$ 

ex) $\Sigma_g = \underbrace{T^2 \# \cdots \# T^2}_{g}$ genus g surf