10. Vector Bundles

Recall: Tangent bundle TM

- $M = \bigcup \mathcal{U}_\alpha$ coord charts on M
- $\bigcup \pi^{-1}(\mathcal{U}_\alpha)$ coord charts on TM with: $\pi^{-1}(\mathcal{U}_\alpha) = \mathcal{U}_\alpha \times \mathbb{R}^n$
  (x^i, v^i)
- On $\pi^{-1}(\mathcal{U}_\alpha) \cap \pi^{-1}(\mathcal{U}_\beta)$, have coord change:
  $(x^i, v^i) \quad (\tilde{x}^i, \tilde{v}^i)$
  $\tilde{x}^i = f^i(x^1, \dots, x^n)$ coords on M
  $\tilde{v}^i = \frac{\partial \tilde{x}^i}{\partial x^k} v^k$.

Idea of vector bundle:

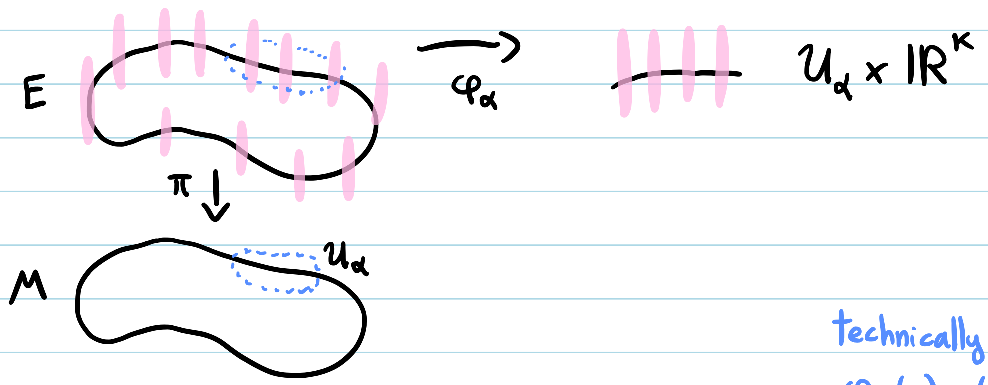
generalize coord change to:

$\tilde{x}^i = f^i(x^1, \dots, x^n)$

$\tilde{v}^i = \gamma(x)^i{}_p \, v^p$ general $k \times k$ matrix $\gamma \in GL(k, \mathbb{R})$

Defn A: $\pi : E \to M$ is a vector bundle of rank k if:

- $\pi : E \to M$ is a surjective map of manifolds with $\pi^{-1}(p) := E_p$ a vector space $\forall p \in M$.
- $\exists$ open cover
  $$M = \bigcup_\alpha \mathcal{U}_\alpha, \quad E = \bigcup_\alpha \pi^{-1}(\mathcal{U}_\alpha)$$
  with diffeos
  $$\varphi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times \mathbb{R}^k$$
  s.t. a) $\pi \circ \varphi_\alpha^{-1}(p, v) = p$
  b) $\varphi_\alpha : E_p \to \{p\} \times \mathbb{R}^k$ isomorphism.



Coords on mfd E: $\pi^{-1}(\mathcal{U}_\alpha)$ are charts

$\varphi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times \mathbb{R}^k$

$e \mapsto (\underbrace{x^1, \dots, x^n}_{\text{coords on } M^n}, \underbrace{v^1, \dots, v^k}_{\text{along } \mathbb{R}^k}) = \varphi_\alpha(e)$

technically: $\psi_\alpha : \mathcal{U}_\alpha \to \psi_\alpha(\mathcal{U}_\alpha)$ chart on M^n

$\varphi_\alpha(e) = (p, v) \in \mathcal{U}_\alpha \times \mathbb{R}^k$

$\Phi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \psi_\alpha(\mathcal{U}_\alpha) \times \mathbb{R}^k \subseteq \mathbb{R}^{n+k}$

$\Phi_\alpha(e) = (\psi_\alpha(p), v) = (x, v) \in \mathbb{R}^{n+k}$ chart on E

Transition functions:

Consider $\varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$

$$(p, v) \mapsto (p, \gamma_{\alpha\beta}(p) v).$$

Call $\gamma_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ transition functions.

Coord change: $(\tilde{x}, \tilde{v}) = (f(x), \gamma_{\alpha\beta}(x) v) = \varphi_\alpha \circ \varphi_\beta^{-1}(x, v)$, (x,v) on $\pi^{-1}(U_\beta)$, $(\tilde{x}, \tilde{v})$ on $\pi^{-1}(U_\alpha)$

Note: $\gamma_{\alpha\beta} \gamma_{\beta\gamma} = \gamma_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. $\varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\beta \circ \varphi_\gamma^{-1} = \varphi_\alpha \circ \varphi_\gamma^{-1}$

Defn B: Cover $M = \bigcup U_\alpha$ by coord charts.
Give matrix-valued smooth functions on overlaps

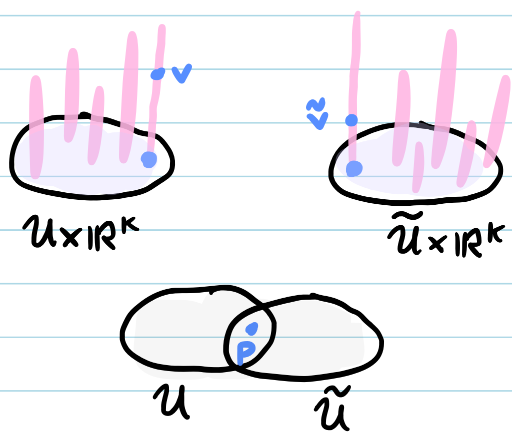
$$\gamma_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$$

s.t. $\gamma_{\alpha\beta} \gamma_{\beta\gamma} = \gamma_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

Define: $E = \left(\bigcup_\alpha U_\alpha \times \mathbb{R}^k \right) / \sim$

where: for $(p, v) \in U_\alpha \times \mathbb{R}^k$
$(p, \tilde{v}) \in U_\beta \times \mathbb{R}^k$

$$(p, v) \sim (p, \tilde{v}) \iff \tilde{v} = \gamma_{\beta\alpha} v$$



Going between defn A ⇝ B

(A) ⇒ (B). Already explained how to get transition functions $\gamma_{\alpha\beta}$ from trivializations $\varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^k$.

$$\varphi_\alpha \circ \varphi_\beta^{-1}(x, v) = (x, \gamma_{\alpha\beta}(x) v)$$

(B) ⇒ (A) $[(p, v, \alpha)] \in E = \bigcup (U_\alpha \times \mathbb{R}^k) / \sim$

$p \in U_\alpha$

projection: $\pi : E \to M$

$$\pi([p, v, \alpha]) = p$$

$E_p = \pi^{-1}(p)$ is vector space: $[(p,u,\alpha)] \in \mathcal{U}_\alpha \times \mathbb{R}^k$
$[(p,v,\alpha)]$

$a[(p,u,\alpha)] + b[(p,v,\alpha)] = [(p, au+bv, \alpha)]$.

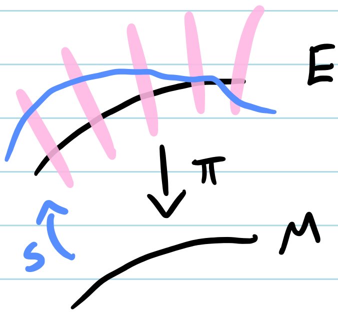
$$\varphi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times \mathbb{R}^k$$
$$[(p,v,\alpha)] \mapsto (p,v).$$

Sections: $s: M \to E$ is a section if: $s(x) \in \pi^{-1}(x) \quad \forall x \in M$.

Over $\mathcal{U}_\alpha$: $\quad s\big|_{\mathcal{U}_\alpha} := \varphi_\alpha \circ s = (x, s_\alpha(x))$ for some smooth vector function $s_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^k$

On overlaps $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$: $\quad s_\alpha = \gamma_{\alpha\beta}\, s_\beta$

$$\varphi_\alpha \circ s = \varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\beta \circ s$$



Notation: $s \in \Gamma(E)$ means s is a section of $E \to M$.

Notation: Write $s = \{s_\alpha\}$ where $s_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^k$ and $s_\alpha = \gamma_{\alpha\beta}\, s_\beta$.

ex) $E = TM$ tangent bundle

$$\varphi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times \mathbb{R}^n$$

$$\varphi_\alpha\left(v^i \frac{\partial}{\partial x^i}\Big|_p \right) = (p, v^1, \dots, v^n).$$

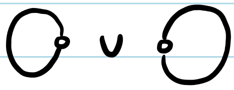
$\gamma_{\beta\alpha} = \left[\dfrac{\partial \tilde{x}}{\partial x}\right]_{n\times n}$ $\quad (\mathcal{U}_\alpha, x), (\mathcal{U}_\beta, \tilde{x})$ overlap of coord charts

Section of TM: vector fields $V \in \Gamma(TM)$

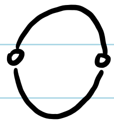
$$V\big|_{\mathcal{U}_\alpha} = (x, V^1(x), \dots, V^n(x)) \quad \text{with} \quad \tilde{V}^i = \frac{\partial \tilde{x}^i}{\partial x^p} V^p.$$

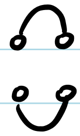
" $s_\alpha = \gamma_{\alpha\beta}\, s_\beta$ "

ex) Möbius bundle $E \to S^1$ (rank 1)

$S^1 =$ 

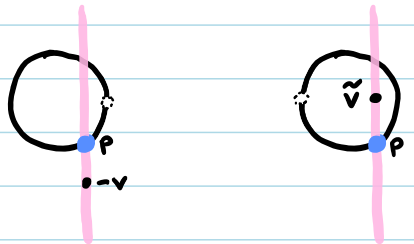
$\mathcal{U} \quad \tilde{\mathcal{U}}$



$\mathcal{U} \cap \tilde{\mathcal{U}}$ $\xrightarrow[\gamma_{12}]{}$ $\begin{cases} +1 & \text{on} \\ -1 & \text{on} \end{cases}$ 

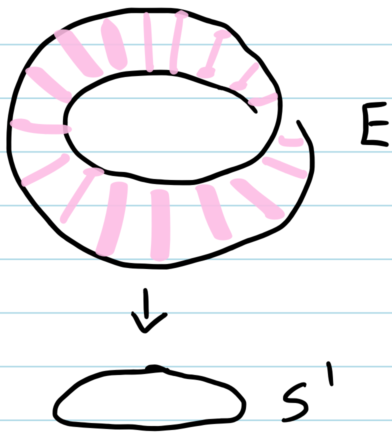
$\pm 1 \in GL(1)$

$E = (\mathcal{U} \times \mathbb{R}) \sqcup (\tilde{\mathcal{U}} \times \mathbb{R}) / \sim$



$(p, v) \sim (p, \tilde{v})$

$\Leftrightarrow v = \gamma_{12} \tilde{v}$



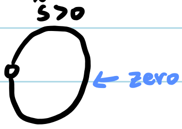
Note: any section of Möbius bundle must pass through zero.

Let $s: S^1 \to E$, $\quad s|_{\mathcal{U}} = (\theta, s(\theta))$

$\quad s|_{\tilde{\mathcal{U}}} = (\tilde{\theta}, \tilde{s}(\tilde{\theta}))$

If e.g. $s(\theta) > 0 \Rightarrow \tilde{s} = \gamma_{12} s \Rightarrow$ $\tilde{s} > 0$ on $\cap$

$\tilde{s} < 0$ on $\cup$



Def: A local frame for E over $\mathcal{U}$ is a basis of sections $(e_1, \dots, e_k)$ over $\mathcal{U}$. I.e. $e_i : \mathcal{U} \to E$

$\{e_1(p), \dots, e_k(p)\}$ basis for E_p $\forall p \in \mathcal{U}$.

ex) TM on $\mathbb{R}^2$. $\mathcal{U} = \mathbb{R}^2 \setminus \{0\}$

$e_1 = x\,\partial_x + y\,\partial_y$

$e_2 = -y\,\partial_x + x\,\partial_y$

local frame over $\mathcal{U}$

ex) $E \to S^1$ Möbius bundle.

Cannot have frame $e_1 \in \Gamma(E)$ over all of S^1. But have local frames e_1 over $\mathcal{U} \subsetneq S^1$.

Dual Bundle:

Recall: Let V be a vector space, $\dim V = n$.
Let $\{e_1, \dots, e_n\}$ be a basis for V.

$V^* =$ linear functionals on V
$= \{ \omega : V \to \mathbb{R} \ : \ \omega \text{ is linear map} \}$

dual basis: $\{e^1, \dots, e^n\}$ defined by: $e^i(e_k) = \delta^i{}_k$.

Can write: $u \in V$ as $u = u^i e_i$
$\omega \in V^*$ as $\omega = \omega_i e^i$

Def: Let $E \to M$ be bundle with: trans fun $\gamma_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to GL(k, \mathbb{R})$

The dual bundle $E^* \to M$
defined by: trans fun $(\gamma_{\alpha\beta}{}^T)^{-1} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to GL(k, \mathbb{R})$

Point of this: sections s of E / ϕ of E^* may be paired together.

$\phi(s) = \phi_\alpha{}^T s_\alpha$ over $\mathcal{U}_\alpha$.

$s\big|_{\mathcal{U}_\alpha} = (x, s_\alpha(x))$ $\qquad s_\alpha = \gamma_{\alpha\beta} s_\beta$

$\phi\big|_{\mathcal{U}_\alpha} = (x, \phi_\alpha(x))$ $\qquad \phi_\alpha = (\gamma_{\alpha\beta}{}^T)^{-1} \phi_\beta$

Note: $\phi_\alpha{}^T s_\alpha = ((\gamma_{\alpha\beta}{}^T)^{-1} \phi_\beta)^T (\gamma_{\alpha\beta} s_\beta) = \phi_\beta{}^T s_\beta$

$\therefore \phi(s) : M \to \mathbb{R}$ well-defined function.

Point of view of local frames:

- $\{e_i\}$ local frame for E
- $\{e^i\}$ local frame for E^* : $e^i(p) \in E_p$ is dual vector to $e_i(p)$.
  $e^i(e_k) = \delta^i{}_k$

- $s(p) = s^i(p)\, e_i(p)$ local section of E
- $\phi(p) = \phi_i(p)\, e^i(p)$ local section of E^*

$$\begin{aligned}\phi(s) &= s^i e_i\, (\phi_k e^k) \\ &= s^i \phi_k\, e_i(e^k) \\ &= s^i \phi_k\, \delta_i^{\ k} \\ &= s^i \phi_i\end{aligned}$$

Isomorphism of bundles: $E \to M$, $E' \to M$, E and E' are isomorphic if:

$\exists$ smooth map $F : E \to E'$

s.t. 1) $\pi' \circ F = \pi$

2) $F\big|_{E_p} : E_p \to E'_p$ is an invertible linear map.