11. Cotangent Bundle

- $TM \to M$ tangent bundle
- $T^*M \to M$ cotangent bundle = dual bundle to TM.

Given a coord chart $(\mathcal{U}, x^i)$:

- $\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}$ local frame for TM

- $\{dx^1, \dots, dx^n\}$ local frame for T^*M

$$dx^i \left(\frac{\partial}{\partial x^k} \right) = \delta^i{}_k .$$

- vector field: $V = \underbrace{V^i}_{\text{components}} \underbrace{\frac{\partial}{\partial x^i}}_{\text{frame}}$

$$\tilde{V}^i = \frac{\partial \tilde{x}^i}{\partial x^p} V^p$$

$$\frac{\partial}{\partial \tilde{x}^i} = \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial}{\partial x^p}$$

$\Rightarrow \tilde{V}^i \frac{\partial}{\partial \tilde{x}^i} = V^i \frac{\partial}{\partial x^i}$ on overlaps $\qquad \frac{\partial \tilde{x}^i}{\partial x^p} \frac{\partial x^p}{\partial \tilde{x}^j} = \frac{\partial \tilde{x}^i}{\partial \tilde{x}^j} = \delta^i{}_j$

- covector field: usually called 1-form: $\omega = \underbrace{\omega_i}_{\text{components}} \underbrace{dx^i}_{\text{frame}}$

$$\tilde{\omega}_i = \frac{\partial x^p}{\partial \tilde{x}^i} \omega_p$$

$$d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^p} dx^p$$

$\Rightarrow \tilde{\omega}_i \, d\tilde{x}^i = \omega_i \, dx^i$ on overlaps

- Pairing

$\omega(V) = \omega_i V^i$ in components

$\omega(V) = (\omega_i dx^i)(V^k \partial_k)$ in local frame

$\quad = \omega_i V^k dx^i(\partial_k) = \omega_i V^i$

check: $\tilde{\omega}_i \tilde{X}^i = \omega_i X^i$ so the pairing is well-defn on overlaps.

<u>Def</u>: From $f \in C^\infty(M)$, can define 1-form df via:

$$df = \frac{\partial f}{\partial x^i} dx^i \quad \text{in coord chart } (\mathcal{U}, x^i).$$

<u>well-defined</u>: Need $\begin{matrix} (df)_i \, dx^i \\ (\widetilde{df})_i \, d\tilde{x}^i \end{matrix}$ on overlap to satisfy

$$(\widetilde{df})_i = \frac{\partial x^p}{\partial \tilde{x}^i} (df)_p.$$

chain rule

$$(\widetilde{df})_i = \frac{\partial f}{\partial \tilde{x}^i} = \frac{\partial f}{\partial x^p} \frac{\partial x^p}{\partial \tilde{x}^i} = \frac{\partial x^p}{\partial \tilde{x}^i} (df)_p \quad \checkmark$$

<u>Pairing</u>: $df(V) = V(f)$

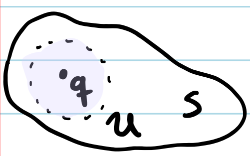
$$df(v) = v^i \frac{\partial f}{\partial x^i}$$

ex) $f(x,y) = x \sin y$
$df = (\sin y) dx + (x \cos y) dy$

<u>Prop</u>: Let M be a connected mfd, $f \in C^\infty(M)$.
If $df \equiv 0$, then $f \equiv \text{const}$.

<u>Pf</u>: Let $p \in M$.
$S = \{ x \in M : f(x) = f(p) \}$

0. S is non-empty
1. $S = f^{-1}(c)$ is closed by continuity of f.
$c = f(p)$
2. S is open: if $q \in S$, let $(\mathcal{U}, x^i)$ be coord chart containing q.
$df = \frac{\partial f}{\partial x^i} dx^i = 0 \implies \frac{\partial f}{\partial x^i} \equiv 0 \ \forall i$ in $\mathcal{U}$.



$\implies f \equiv \text{const}$ in $\mathcal{U}$ $\quad f \equiv f(q) = f(p)$
$\implies \mathcal{U} \subseteq S$

3. S is clopen. $\square$

Pullback: Let $F: M \to N$ smooth map.

Recall differential: $dF_p : T_pM \to T_{F(p)}N$

$$dF_p(V) = \frac{d}{dt}\Big|_{t=0} F\circ\gamma(t), \qquad \begin{matrix}\gamma(0)=p\\ \dot\gamma(0)=V\end{matrix}$$

Pullback: Given ω 1-form on N, define $F^*\omega$ 1-form on M by pairing:

$$(F^*\omega)_p(X) = \omega_{F(p)}(dF_p(X)) \qquad \forall X \in T_pM.$$

In coords: $\omega = \omega_\alpha(y)\, dy^\alpha$ $\qquad$ y^α coords on N

$$F^*\omega = \frac{\partial F^\alpha}{\partial x^i}(x)\, \omega_\alpha(F(x))\, dx^i \qquad x^i \text{ coords on } M$$

check:

$$(F^*\omega)_p(X) = \frac{\partial F^\alpha}{\partial x^i}(p)\, \omega_\alpha(F(p))\, X^i, \qquad X = X^i\partial_i$$

$$= \omega_\alpha(F(p))\, \underbrace{\frac{\partial F^\alpha}{\partial x^i}(p)\, X^i}_{} = \omega_{F(p)}(dF_p(X)).$$

$\omega_\alpha \quad (dF_p(X))^\alpha$

Prop: $F^*(u\omega) = (u\circ F)\, F^*\omega$, $\quad \forall u \in C^\infty(N)$, ω 1-form on N

Pf: $F^*(u\omega) = F^*(u\,\omega_\alpha\, dy^\alpha)$

$$= \frac{\partial F^\alpha}{\partial x^i}\, u(F(x))\, \omega_\alpha(F(x))\, dx^i$$

$$= (u\circ F)\, F^*\omega. \quad \square$$

Prop: $F^*du = d(u\circ F) \qquad \forall u \in C^\infty(N)$

Pf: $F^*du = F^*(\partial_\alpha u\, dy^\alpha)$

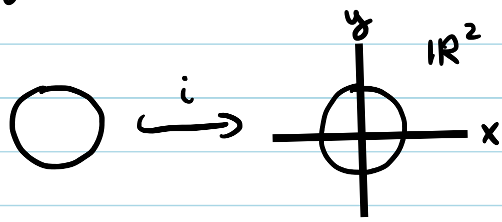
$$= \frac{\partial F^\alpha}{\partial x^i}(x)\, \frac{\partial u}{\partial y^\alpha}(F(x))\, dx^i$$

$$d(u\circ F) = \frac{\partial}{\partial x^i}(u(F(x)))\, dx^i = \frac{\partial u}{\partial y^\alpha}(F(x))\, \frac{\partial F^\alpha}{\partial x^i}(x)\, dx^i \quad \square$$

Note: $F^*(\omega_\alpha \, dy^\alpha) = (\omega_\alpha \circ F) \, d(y^\alpha \circ F)$.

ex) $i : S^1 \to \mathbb{R}^2$

$i(e^{i\theta}) = (\cos\theta, \sin\theta)$

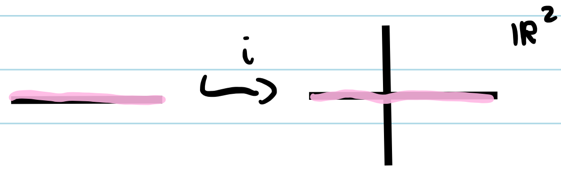


$i^*(x\,dy - y\,dx) = ?$

$$
\begin{aligned}
&= (x \circ i) \, d(y \circ i) - (y \circ i) \, d(x \circ i) \\
&= \cos\theta \, d\sin\theta - \sin\theta \, d\cos\theta \\
&= \cos^2\theta \, d\theta + \sin^2\theta \, d\theta \\
&= d\theta.
\end{aligned}
$$

ex) $i : L \to \mathbb{R}^2$

$i(x) = (x, 0)$



$i^* dx = dx$

$i^* dy = 0$

Prop: If $F : M \to N$

$G : N \to Z$

then $(G \circ F)^* = F^* \circ G^*$

Pf:

$$
\begin{aligned}
(G \circ F)^* \omega \, (X) &= \omega \left(d(G \circ F) X \right) \\
&= \omega (dG \, dF \, X) \qquad \text{chain rule} \\
&= G^* \omega (dF \, X) \\
&= F^* G^* \omega \, (X).
\end{aligned}
$$

$\square$