13. Riemannian Metrics

Def: A Riemannian metric is a tensor field $g \in \Gamma(T^*M \otimes T^*M)$ s.t. $\forall$ coord charts $(\mathcal{U}, x^i)$,

$$g = g_{ij}\, dx^i \otimes dx^j$$ with: (g_{ij}) a symmetric positive definite matrix at all points in $\mathcal{U}$.

Notation: $dx^i dx^j = \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i)$

$$g = g_{ij}\, dx^i dx^j.$$

Inner Product: $g(V,W) = g_{ij}(p)\, V^i W^j \quad \forall V, W \in T_pM.$

Using frames:
$$\begin{aligned} g(V,W) &= g_{ij}\, dx^i \otimes dx^j (V^k \partial_k, W^l \partial_l) \\ &= g_{ij} V^k W^l\, dx^i \otimes dx^j (\partial_k, \partial_l) \\ &= g_{ij} V^k W^l\, dx^i(\partial_k)\, dx^j(\partial_l) \\ &= g_{ij} V^i W^j \end{aligned}$$

Note: $g_{ij} = g(\partial_i, \partial_j)$

Change of coords: On overlap $(\mathcal{U}, x^i)$, $(\tilde{\mathcal{U}}, \tilde{x}^i)$

$$\tilde{g}_{ij} = \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial x^q}{\partial \tilde{x}^j} g_{pq}$$

ex) Euclidean metric g_{EUC} on $\mathbb{R}^n$.

$$g_{EUC} = \delta_{ij}\, dx^i dx^j = dx^1 dx^1 + \cdots + dx^n dx^n$$

$$g_{EUC}(V,W) = \delta_{ij} V^i W^j = \sum_{i=1}^n V^i W^i = V \cdot W.$$

ex) g_{EUC} on $\mathbb{R}^2$. Change of coords to polar:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\underbrace{\tilde{g}_{ij}}_{g_{Euc}\text{ in polar}} = \frac{\partial x^p}{\partial \tilde{x}^i} \underbrace{\delta_{pq}}_{g_{Euc}\text{ in Cartesian}} \frac{\partial x^q}{\partial \tilde{x}^j} \qquad \begin{aligned}(x^1, x^2) &= (x, y)\\ (\tilde{x}^1, \tilde{x}^2) &= (r, \theta)\end{aligned}$$

$$\tilde{g}_{rr} = \cos^2\theta + \sin^2\theta = 1$$
$$\tilde{g}_{r\theta} = \cos\theta(-r\sin\theta) + \sin\theta(r\cos\theta) = 0$$
$$\tilde{g}_{\theta\theta} = (-r\sin\theta)^2 + (r\cos\theta)^2 = r^2$$

$$g_{Euc} = dr\,dr + r^2\,d\theta\,d\theta \quad \text{on} \quad \mathbb{R}^2 \setminus \{0\}$$

$$g_{Euc} = dx\,dx + dy\,dy$$

Prop: Every manifold admits a Riemannian metric.

Pf: A) Let: $M = \bigcup \mathcal{U}_\alpha$ cover by coord charts

: $\{\psi_\alpha\}$ partion of unity subordinate to $\{\mathcal{U}_\alpha\}$.

Def: A partition of unity subordinate to $\{\mathcal{U}_\alpha\}$ is a collection of smooth functions $\psi_\alpha : \mathcal{U}_\alpha \to [0,1]$ s.t.

1) $\operatorname{supp} \psi_\alpha \subset\subset \mathcal{U}_\alpha \quad \forall \alpha$
2) $\forall p \in M \; \exists$ nbhd V s.t. $V \cap \operatorname{supp} \psi_\alpha \neq \phi$ for only finitely many α
3) $\sum_\alpha \psi_\alpha(x) = 1 \quad \forall x \in M$.

Thm: Let M be a smooth mfd, $M = \bigcup \mathcal{U}_\alpha$ open cover.
$\exists \{\psi_\alpha\}$ partition of unity subordinate to $\mathcal{U}_\alpha$.

Proof: [Lee Theorem 2.23]

B) Denote coords attached to $\mathcal{U}_\alpha$ by $(x_\alpha^1, \dots, x_\alpha^n)$.
On $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$, $x_\alpha^i(x_\beta^1, \dots, x_\beta^n)$ change of coords.
Define over $\mathcal{U}_\alpha$:

$$g|_{\mathcal{U}_\alpha} = (g_\alpha)_{ij}\, dx_\alpha^i \otimes dx_\alpha^j, \text{ where:}$$

$$(g_\alpha)_{ij} = \sum_\mu \psi_\mu \sum_p \frac{\partial x_\mu^p}{\partial x_\alpha^i} \frac{\partial x_\mu^p}{\partial x_\alpha^j}$$

c) Check $(g_\alpha)_{ij}$ is symmetric positive-definite and:

$$(g_\beta)_{ij} = \frac{\partial x_\alpha^p}{\partial x_\beta^i} \frac{\partial x_\alpha^q}{\partial x_\beta^j} (g_\alpha)_{pq}.$$

$$\sum \frac{\partial x_\alpha^p}{\partial x_\beta^i} \frac{\partial x_\alpha^q}{\partial x_\beta^j} \underbrace{\left(\psi_\mu \frac{\partial x_\mu^r}{\partial x_\alpha^p} \frac{\partial x_\mu^r}{\partial x_\alpha^q}\right)}_{(g_\alpha)_{pq}} = \sum \psi_\mu \frac{\partial x_\mu^r}{\partial x_\beta^i} \frac{\partial x_\mu^r}{\partial x_\beta^j} = (g_\beta)_{ij} \qquad \square$$

Norms: Let g be a metric on M.

- $g(\cdot,\cdot)$ inner product on T_pM
- $|v|_g = (g_{ij} v^i v^j)^{\frac{1}{2}}$ norm on T_pM
- Inverse notation: g^{ij} is inverse of g_{ij} : $g^{ik} g_{kj} = \delta^i{}_k$.
- Inner product on T^*M:

$$g(\alpha,\beta) = g^{ij} \alpha_i \beta_j \qquad \forall \text{ 1-forms } \alpha = \alpha_i dx^i$$
$$|\alpha|_g = (g^{ij} \alpha_i \alpha_j)^{\frac{1}{2}} \qquad \beta = \beta_i dx^i$$

Exercise: 1) $\tilde{g}^{ij} = \frac{\partial \tilde{x}^i}{\partial x^p} \frac{\partial \tilde{x}^j}{\partial x^q} g^{pq}$

2) $\tilde{g}^{ij} \tilde{\alpha}_i \tilde{\beta}_j = g^{ij} \alpha_i \beta_j$ so that $g(\alpha,\beta)$ is well-defined.

Raising/Lowering the index: Let g metric
V vector field
α 1-form

$$\alpha = \alpha_i dx^i \rightsquigarrow \alpha^\# = \alpha^i \partial_i$$
$$\alpha^i = g^{ik} \alpha_k$$

$$V = V^i \partial_i \rightsquigarrow V^\flat = V_i dx^i$$
$$V_i = g_{ik} V^k$$

check: $\alpha^\# \in \Gamma(TM)$ e.g. $\tilde{\alpha}^i = \frac{\partial \tilde{x}^i}{\partial x^p} \alpha^p$.
$V^\flat \in \Gamma(T^*M)$

Without coords:

$\alpha^\#(f) = g(\alpha, df)$ action of VF $\alpha^\#$ on functions

$V^\flat(X) = g(V, X)$ dual linear functional to V

Pullback Metric: $F: M \to (N, g)$ with $\operatorname{Ker} dF_p = \{0\}$ $\forall p \in M$.
$F^* g$ metric on M given by:

$$(F^*g)_{ij}(p) = g_{\alpha\beta}(F(p)) \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j}$$

y^α coords on N, $F^\alpha = y^\alpha \circ F$
x^i coords on M

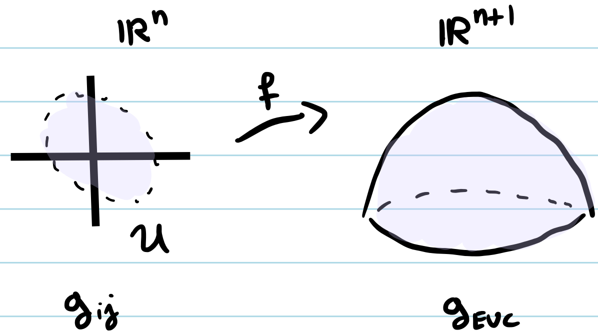
without coords: $(F^*g)_p(X, Y) = g_{F(p)}(dF_p(X), dF_p(Y))$.

Def: An isometry is a diffeo $F: (M, \tilde{g}) \to (N, g)$ s.t. $F^* g = \tilde{g}$.

Parametrized Hypersurfaces

Let $f: \underset{\subseteq \mathbb{R}^n}{\mathcal{U}} \to \mathbb{R}^{n+1}$ be smooth map with $\operatorname{Ker} df_p = \{0\}$ $\forall p \in \mathcal{U}$.

The classical metric tensor (AKA 1st fund form) is:



$$g = f^* g_{Euc}.$$

$$g_{ij} = \left\langle \frac{\partial f}{\partial x^i}, \frac{\partial f}{\partial x^j} \right\rangle_{g_{Euc}}$$

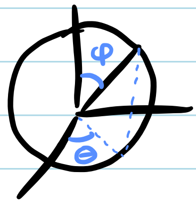
$$g_{ij} = \delta_{\alpha\beta} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j}$$

$$g = (df)^T (df)$$

$$\overset{n=2}{=} \begin{pmatrix} \partial_1 f^1 & \partial_1 f^2 & \partial_1 f^3 \\ \partial_2 f^1 & \partial_2 f^2 & \partial_2 f^3 \end{pmatrix} \begin{pmatrix} \partial_1 f^1 & \partial_2 f^1 \\ \partial_1 f^2 & \partial_2 f^2 \\ \partial_1 f^3 & \partial_2 f^3 \end{pmatrix}$$

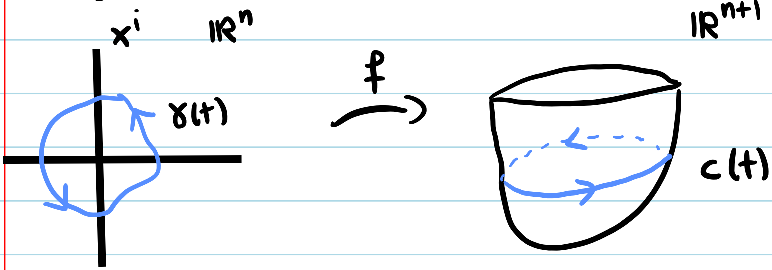
ex) Spherical coords

$f(\theta, \varphi) = (\cos\theta \sin\varphi, \ \sin\theta \sin\varphi, \ \cos\varphi)$



check: $g = (\sin^2\varphi)\, d\theta\, d\theta + d\varphi\, d\varphi$

Length of curves:



$c(t) = f \circ \gamma(t)$

$\gamma : [a,b] \to \mathcal{U}$ given

$c : [a,b] \to \mathbb{R}^{n+1}$ $\quad c = f \circ \gamma$

$L = \int_a^b |\dot{c}|\, dt$ length of curve in $\mathbb{R}^{n+1}$. How to write in coords on $\mathcal{U} \subseteq \mathbb{R}^n$?

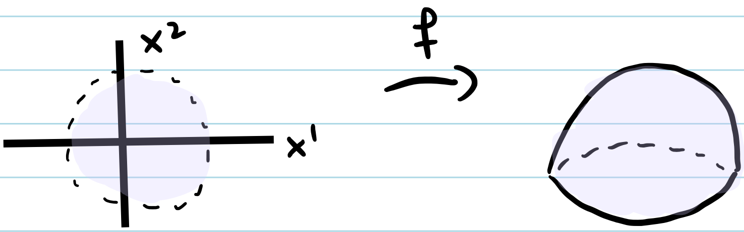
$\dot{c} = \frac{\partial f}{\partial x^i} \frac{d\gamma^i}{dt}$, $\quad \gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$

dot prod in $\mathbb{R}^{n+1}$

$\dot{c} \cdot \dot{c} = \left\langle \frac{\partial f}{\partial x^i}, \frac{\partial f}{\partial x^j} \right\rangle_{g_{Euc}} \dot{\gamma}^i \dot{\gamma}^j = g_{ij} \dot{\gamma}^i \dot{\gamma}^j = |\dot{\gamma}|_g^2$

$\therefore L = \int_a^b |\dot{\gamma}|_g \, dt$ expression for length in coords $\mathcal{U} \subseteq \mathbb{R}^n$.

Surface area: $S \subseteq \mathbb{R}^3$ surface in 3d, $f : \mathcal{U} \to \mathbb{R}^3$



$S \subseteq \mathbb{R}^3$

Formula from Calc IV: $dA = \left| \frac{\partial f}{\partial x^1} \times \frac{\partial f}{\partial x^2} \right| dx^1 dx^2$

Exercise: show $\left|\frac{\partial f}{\partial x^1} \times \frac{\partial f}{\partial x^2}\right| = \sqrt{\det g_{ij}}$.

$\therefore \; dA = \sqrt{\det g_{ij}}\, dx^1 dx^2$

$$\text{Area}(S) = \iint_{\mathcal{U}} \sqrt{\det g_{ij}}\, dx^1 dx^2.$$

In higher dim: $f: \underset{\subseteq \mathbb{R}^n}{\mathcal{U}} \to \mathbb{R}^{n+1}$ parametrized hypersurface,

define: $\text{Area}(S) = \int_{\mathcal{U}} \sqrt{\det g_{ij}}\, dx^1 \cdots dx^n.$

ex) Spherical coords

$f: (0,2\pi) \times (0,\pi) \to \mathbb{R}^3$

$f(\theta, \varphi) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)$

$g = (\sin^2\varphi)\, d\theta d\theta + d\varphi d\varphi$

$dA = \sqrt{\det g}\, d\theta d\varphi$

$dA = (\sin^2\varphi)\, d\theta d\varphi$ formula to compute the area of S^2.

$$\text{Area}(S^2) = \int_0^{2\pi} \int_0^{\pi} (\sin^2\varphi)\, d\varphi\, d\theta = 4\pi.$$