14. Differential Forms

Def: A differential form of order k, denoted $\omega \in \Omega^k(M)$, is a section $\omega \in \Gamma(T^*M^{\otimes k})$ which is totally anti-symmetric.

$$\omega(V_1, \dots, V_k) = \omega_{i_1 \cdots i_k} V_1^{i_1} \cdots V_k^{i_k} \quad \text{with } \omega_{i_1 \cdots i_k} \text{ antisymmetric in } i_1 \cdots i_k.$$

ex) $\omega \in \Omega^2(M)$

$\omega(V, W) = \omega_{ij} V^i W^j$, with $\omega_{ij} = -\omega_{ji}$.

$\Rightarrow \omega(V,W) = -\omega(W,V)$.

Notation: $dx^{i_1} \wedge \dots \wedge dx^{i_k} = \sum\limits_{P \in S_k} \operatorname{sgn}(P)\, dx^{i_{P(1)}} \otimes \dots \otimes dx^{i_{P(k)}}$

ex) $dx^1 \wedge dx^2 = dx^1 \otimes dx^2 - dx^2 \otimes dx^1$

$$dx^1 \wedge dx^2 (Y, Z) = dx^1 \otimes dx^2 (Y, Z) - dx^2 \otimes dx^1 (Y, Z)$$

$$= Y^1 Z^2 - Y^2 Z^1.$$

Note: $dx^1 \wedge dx^2 = -dx^2 \wedge dx^1$

Note: $dx^i \wedge dx^i = 0$

Note: $dx^1 \wedge \dots \wedge dx^k (V_1, \dots, V_k) = \det\left(dx^i(V_j)\right) = \det\left(V^i{}_j\right)_{k \times k}$

Notation: $\omega \in \Omega^k(M)$

$$\omega = \frac{1}{k!} \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

ex) $\omega \in \Omega^2(M)$

$\omega = \frac{1}{2} \omega_{ij} dx^i \wedge dx^j$. Why ½?

$$\omega(V, W) = \frac{1}{2} \omega_{ij} dx^i \wedge dx^j (V, W)$$

$$= \frac{1}{2} \omega_{ij} (V^i W^j - V^j W^i)$$

$$= \omega_{ij} V^i W^j \quad \text{since } \omega_{ij} = -\omega_{ji}.$$

$$\omega(\partial_i, \partial_j) = \omega_{ij}$$

Wedge Product: $\omega \in \Omega^k(M)$
$\gamma \in \Omega^\ell(M)$

$$\omega \wedge \gamma = \frac{1}{k!}\frac{1}{\ell!} \omega_{i_1 \cdots i_k} \gamma_{j_1 \cdots j_\ell} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_\ell}$$

ex) $\omega, \gamma \in \Omega^1(M)$.

$$\begin{aligned} \omega \wedge \gamma &= \omega_i \gamma_k \, dx^i \wedge dx^k \\ &= \frac{1}{2}(\omega_i \gamma_k - \omega_k \gamma_i) \, dx^i \wedge dx^k \\ &= \frac{1}{2}(\omega \wedge \gamma)_{ik} \, dx^i \wedge dx^k \end{aligned}$$

$$\Rightarrow (\omega \wedge \gamma)_{ik} = \omega_i \gamma_k - \omega_k \gamma_i$$

$$\begin{aligned} (\omega \wedge \gamma)(V, W) &= (\omega \wedge \gamma)_{ik} V^i W^k \\ &= \omega_i \gamma_k V^i W^k - \omega_k \gamma_i V^i W^k \end{aligned}$$

$$\Rightarrow (\omega \wedge \gamma)(V, W) = \omega(V)\gamma(W) - \omega(W)\gamma(V).$$

Note: $\omega \wedge \gamma = (-1)^{k\ell} \gamma \wedge \omega$, $\quad \omega \in \Omega^k(M)$, $\gamma \in \Omega^\ell(M)$

Note: $\gamma \wedge \gamma = 0$ if $\gamma \in \Omega^k(M)$ with k odd.

ex) $\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$ on $\mathbb{R}^4$
$\omega \wedge \omega = 2\, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$

Pullback: $F: M \to N$, $\omega \in \Omega^k(N)$.
$F^*\omega \in \Omega^k(M)$ defined by:

$$F^*\omega = \frac{1}{k!} \omega_{i_1 \cdots i_k} \circ F \, d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F)$$

$$F^*\omega(x) = \frac{1}{k!} \omega_{i_1 \cdots i_k}(F(x)) \frac{\partial F^{i_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial F^{i_k}}{\partial x^{\alpha_k}} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}$$

where: x coords on M, y coords on N.

$$(F^*\omega)_p(V_1,\dots,V_k) = \omega_{F(p)}(dF_p(V_1),\dots,dF_p(V_k)) \quad \forall V_i \in T_pM.$$

ex) $\omega = dx \wedge dy$ on $\mathbb{R}^2$

$F(r,\theta) = (r\cos\theta, r\sin\theta)$

$$\begin{aligned} F^*\omega &= d(r\cos\theta)\wedge d(r\sin\theta) \\ &= (\cos\theta\, dr - r\sin\theta\, d\theta)\wedge(\sin\theta\, dr + r\cos\theta\, d\theta) \\ &= r\, dr\wedge d\theta. \end{aligned}$$

Exterior Derivative

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

$$\alpha \overset{loc}{=} \frac{1}{k!}\alpha_{i_1\cdots i_k}\, dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

$$d\alpha \overset{loc}{=} \frac{1}{k!}\partial_p \alpha_{i_1\cdots i_k}\, dx^p \wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

Why is $d\alpha \in \Omega^{k+1}(M)$ well-defined? Check for $\alpha \in \Omega^1(M)$.

$$\begin{aligned} \alpha &= \alpha_i\, dx^i \\ d\alpha &= \partial_k \alpha_i\, dx^k \wedge dx^i \\ &= \frac{1}{2}(\partial_k\alpha_i - \partial_i\alpha_k)\, dx^k\wedge dx^i \end{aligned}$$

$$\Rightarrow (d\alpha)_{ki} = \partial_k\alpha_i - \partial_i\alpha_k$$

Need to show: $(d\alpha)_{ki} = \dfrac{\partial\tilde{x}^p}{\partial x^k}\dfrac{\partial\tilde{x}^q}{\partial x^i}(d\alpha)_{pq}$ on $(\mathcal{U},x)\cap(\tilde{\mathcal{U}},\tilde{x})$

$$\begin{aligned} (d\alpha)_{ki} &= \frac{\partial}{\partial x^k}\alpha_i - \frac{\partial}{\partial x^i}\alpha_k \\ &= \frac{\partial}{\partial x^k}\left(\frac{\partial\tilde{x}^q}{\partial x^i}\tilde{\alpha}_q\right) - \frac{\partial}{\partial x^i}\left(\frac{\partial\tilde{x}^q}{\partial x^k}\tilde{\alpha}_q\right) \\ &= \frac{\partial\tilde{x}^q}{\partial x^i}\frac{\partial}{\partial x^k}\tilde{\alpha}_q(\tilde{x}) - \frac{\partial\tilde{x}^q}{\partial x^k}\frac{\partial}{\partial x^i}\tilde{\alpha}_q(\tilde{x}) \\ &= \frac{\partial\tilde{x}^q}{\partial x^i}\frac{\partial\tilde{x}^p}{\partial x^k}(d\alpha)_{pq} \end{aligned}$$

Key Cancellation! $\partial_i\partial_k\tilde{x} = \partial_k\partial_i\tilde{x}$

$$\frac{\partial}{\partial x^k} = \frac{\partial\tilde{x}^p}{\partial x^k}\frac{\partial}{\partial\tilde{x}^p}$$

An identical calculation shows:

Prop: $f: M \to N$ smooth map. Then $f^* d\alpha = d f^* \alpha \quad \forall \alpha \in \Omega^k(N)$

e.g. check for 1-forms $\alpha \in \Omega^1(N)$.
$\alpha = \alpha_i \, dx^i$

$$(f^* d\alpha)_{ki}(x) = \frac{\partial f^p}{\partial x^k} \frac{\partial f^q}{\partial x^i} (\partial_p \alpha_q - \partial_q \alpha_p)(f(x))$$

$$(d f^* \alpha)_{ki}(x) = \frac{\partial}{\partial x^k} \left(\frac{\partial f^q}{\partial x^i} \alpha_q (f(x)) \right) - (k \leftrightarrow i)$$

Show these are equal by cancelling $\frac{\partial}{\partial x^k} \frac{\partial}{\partial x^i} f = \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^k} f$.

Prop: $d^2 = 0$.

Pf: $d^2 \alpha = \frac{1}{k!} \underbrace{\partial_p \partial_q}_{\text{sym}} \alpha_{i_1 \cdots i_k} \underbrace{dx^p \wedge dx^q}_{\text{anti-sym}} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$

$= 0. \quad \square$

Prop: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta, \qquad \alpha \in \Omega^k(M)$

Pf: $d\left(\frac{1}{k!} \frac{1}{l!} \alpha_{i_1 \cdots i_k} \beta_{j_1 \cdots j_l} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l} \right)$

$= \frac{1}{k! l!} \partial_p \alpha_{i_1 \cdots i_k} \beta_{j_1 \cdots j_l} dx^p \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l}$

$+ \frac{1}{k! l!} \alpha_{i_1 \cdots i_k} \partial_p \beta_{j_1 \cdots j_l} \underbrace{dx^p \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge}_{\text{pick up } (-1)^k} dx^{j_1} \wedge \cdots \wedge dx^{j_l} \quad \square$

ex) $f \in C^\infty(\mathbb{R}^3)$
$\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz \in \Omega^1(\mathbb{R}^3)$
$\beta = \beta_3 dx \wedge dy + \beta_2 dz \wedge dx + \beta_1 dy \wedge dz \in \Omega^2(\mathbb{R}^3)$

$df = (\partial_1 f) dx + (\partial_2 f) dy + (\partial_3 f) dz \quad$ " ∇f "

$$d\alpha = (\partial_1 \alpha_2 - \partial_2 \alpha_1)\, dx \wedge dy - (\partial_1 \alpha_3 - \partial_3 \alpha_1)\, dz \wedge dx + (\partial_2 \alpha_3 - \partial_3 \alpha_2)\, dy \wedge dz$$

"curl α"

$$d\beta = (\partial_1 \beta_1 + \partial_2 \beta_2 + \partial_3 \beta_3)\, dx \wedge dy \wedge dz$$

"div β"

Prop: If $\omega \in \Omega^1(M)$, then:

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

Pf: Recall $\omega = \omega_i\, dx^i$

$$d\omega = \frac{1}{2} (d\omega)_{ij}\, dx^i \wedge dx^j$$

$$(d\omega)_{ij} = \partial_i \omega_j - \partial_j \omega_i$$

$$d\omega(X, Y) = (d\omega)_{ij} X^i Y^j = (\partial_i \omega_j - \partial_j \omega_i) X^i Y^j$$

$$X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$
$$= X^i \partial_i (\omega_j Y^j) - Y^i \partial_i (\omega_j X^j)$$
$$- \omega_i (\underbrace{X^j \partial_j Y^i} - \underbrace{Y^j \partial_j X^i})$$
$$= X^i \partial_i \omega_j Y^j - Y^i \partial_i \omega_j X^j \; \square$$

Lie Derivative

$$L_V \omega = \frac{d}{dt}\Big|_{t=0} \Theta_t^* \omega, \qquad \Theta_t \text{ flow of } V: \frac{d}{dt}\Big|_{t=0} \Theta_t = V.$$

Cartan's Magic Formula

$$L_V \omega = V \lrcorner\, d\omega + d(V \lrcorner\, \omega)$$

where: $V \lrcorner : \Omega^k(M) \to \Omega^{k-1}(M)$

$$(V \lrcorner\, \Upsilon)_{i_1 \cdots i_{k-1}} = V^p \Upsilon_{p\, i_1 \cdots i_{k-1}}$$

$$(V \lrcorner\, \Upsilon)(W_1, \dots, W_{k-1}) = \Upsilon(V, W_1, \dots, W_{k-1})$$

Proof of Cartan's Formula: Let's only prove it for 1-forms.

Let $\omega \in \Omega^1(M)$.

$$L_V \omega = \frac{d}{dt}\bigg|_{t=0} \Theta_t^* \omega$$

$$= \frac{d}{dt}\bigg|_{t=0} \left(\frac{\partial \Theta_t^\alpha}{\partial x^i} \omega_\alpha(\Theta_t(x))\, dx^i \right)$$

$$= (\partial_i V^\alpha \omega_\alpha + \delta^\alpha{}_i\, \partial_\rho \omega_\alpha V^\rho)\, dx^i$$

$$= (\partial_i V^\alpha \omega_\alpha + V^\rho \partial_\rho \omega_i)\, dx^i$$

$$V \lrcorner\, d\omega = V^\rho (d\omega)_{\rho i}\, dx^i$$

$$= V^\rho (\partial_\rho \omega_i - \partial_i \omega_\rho)\, dx^i$$

$$d(V \lrcorner\, \omega) = d(V^\rho \omega_\rho)$$

$$= \partial_i (V^\rho \omega_\rho)\, dx^i$$

$$= (\partial_i V^\rho \omega_\rho + V^\rho \partial_i \omega_\rho)\, dx^i$$

$$\Rightarrow V \lrcorner\, d\omega + d(V \lrcorner\, \omega) \underset{\uparrow \text{ cancellation}}{=} (V^\rho \partial_\rho \omega_i + \partial_i V^\rho \omega_\rho)\, dx^i \quad \checkmark$$

$\square$