15. Orientations

Def: A smooth mfd of dim $= n$ is oriented if

$M = \bigcup_\alpha \mathcal{U}_\alpha$ s.t.

change of coords $\tilde{x}^i = f^i(x)$ on overlaps $\mathcal{U} \cap \tilde{\mathcal{U}}$ satisfy:

$$\det \underbrace{\left(\frac{\partial \tilde{x}^i}{\partial x^j} \right)}_{n \times n \text{ matrix}} > 0 .$$

Def: M is orientable if it can be oriented.

ex) Show S^n is orientable. Can use $S^n = \mathcal{U} \cup \tilde{\mathcal{U}}$ stereographic projection coords

Oriented Frame: Let M be oriented, $p \in M$.
A basis $e_1, \dots, e_n$ for T_pM is positively oriented if:

$$\det(e^i{}_j) > 0, \quad \text{where } e_j = e^i{}_j \frac{\partial}{\partial x^i}.$$

Well-defined: if

$$e_j = e^i{}_j \frac{\partial}{\partial x^i} = \tilde{e}^i{}_j \frac{\partial}{\partial \tilde{x}^i}$$

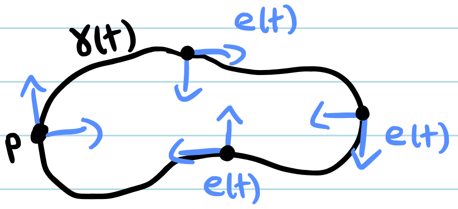
then: $\det(\tilde{e}^i{}_j) \underset{(*)}{=} \det\left(\frac{\partial \tilde{x}^i}{\partial x^p} e^p{}_j \right) = \underbrace{\det\left(\frac{\partial \tilde{x}}{\partial x} \right)}_{>0} \det(e^i{}_j)$

$\therefore \det(e^i{}_j) > 0 \iff \det(\tilde{e}^i{}_j) > 0$

(*) $e^i{}_j \frac{\partial}{\partial x^i} = \underbrace{e^i{}_j \frac{\partial \tilde{x}^p}{\partial x^i}}_{= \tilde{e}^p{}_j} \frac{\partial}{\partial \tilde{x}^p} \Rightarrow \tilde{e}^i{}_j = \frac{\partial \tilde{x}^i}{\partial x^p} e^p{}_j .$

Note: Let M be oriented.
Let $\gamma : [0,1] \to M$ be a curve with $\gamma(0) = \gamma(1) = p$.
Suppose $t \to e_a(t)$ is a continuous moving frame.
($e_a(t)$ is a basis for $T_{\gamma(t)}M$)
Then: if $[e_a(0)]$ is oriented, so is $[e_a(1)]$.

ex)



$\det [e^i_j(t)] > 0$

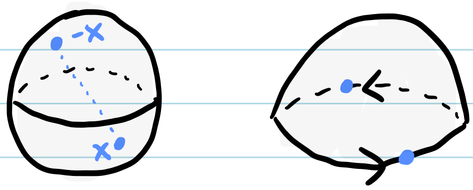
cannot cross zero

ex)

Möbius strip not orientable

Exercise: work this out with equations using e.g. $\tau_{12} = \{ {+1 \atop -1}$ bundle $\to S^1$

ex) $\mathbb{R}\mathbb{P}^2 = S^2/\sim$, $x \sim -x$ is not orientable



Take path: $\gamma(t) = (0, \sin t, \cos t)$

$\gamma(0) = (0,0,1)$

$\gamma(\pi) = (0,0,-1)$

$\gamma(0) = \gamma(\pi)$ on $\mathbb{R}\mathbb{P}^2$.

Moving frame: $e_1(t) = (0, \cos t, -\sin t)$ $\quad$ $e_i(t) \cdot \gamma(t) = 0$

$e_2(t) = (1, 0, 0)$

$\{e_1(0), e_2(0)\} = \{(0,1,0), (1,0,0)\}$

$\{e_1(\pi), e_2(\pi)\} = \{(0,-1,0), (1,0,0)\}$

Coords near $(0,0,1)$: $\varphi(x, y, \sqrt{1-x^2-y^2}) = (x,y)$

$e_1(0) = \frac{\partial}{\partial y}$, $e_2(0) = \frac{\partial}{\partial x}$

$e_1(\pi) = -\frac{\partial}{\partial y}$, $e_2(\pi) = \frac{\partial}{\partial x}$

$\det(e^i_j(0)) < 0$

$\det(e^i_j(\pi)) > 0$

Closed loop reversed orientation $\Rightarrow$ $\mathbb{R}\mathbb{P}^2$ not orientable.

Top forms: Let $\mu \in \Omega^n(M)$, $\dim M = n$.

On $(\mathcal{U}, x)$: $\mu \overset{loc}{=} f(x)\, dx^1 \wedge \cdots \wedge dx^n$. Say $\mu \in \Omega^n(M)$ is nowhere vanishing top-form if $f(x) \neq 0$.

On $(\tilde{\mathcal{U}}, \tilde{x})$: $\mu \overset{loc}{=} \tilde{f}(\tilde{x})\, d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n$

Claim: $\tilde{f} = \det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) f$. $(*)$

check for $n=2$: $\mu = f(x)\, dx^1 \wedge dx^2$
$= \mu_{12}\, dx^1 \wedge dx^2$

Know: $\tilde{\mu}_{12} = \frac{\partial x^p}{\partial \tilde{x}^1} \frac{\partial x^q}{\partial \tilde{x}^2} \mu_{pq}$

$\Rightarrow \tilde{\mu}_{12} = \frac{\partial x^1}{\partial \tilde{x}^1} \frac{\partial x^2}{\partial \tilde{x}^2} \mu_{12} + \frac{\partial x^2}{\partial \tilde{x}^1} \frac{\partial x^1}{\partial \tilde{x}^2} \mu_{21}$

$$\Rightarrow \tilde{\mu}_{12} = \det \begin{pmatrix} \frac{\partial x^1}{\partial \tilde{x}^1} & \frac{\partial x^1}{\partial \tilde{x}^2} \\ \frac{\partial x^2}{\partial \tilde{x}^1} & \frac{\partial x^2}{\partial \tilde{x}^2} \end{pmatrix} \mu_{12}.$$ ✓

Prop: Let M be a smooth manifold of dim n.
M orientable $\Longleftrightarrow$ $\exists$ smooth nowhere vanishing top form $\omega \in \Omega^n(M)$.

Pf: $(\Rightarrow)$ Suppose M is orientable.

1. Equip M with a metric g_{ij}.

2. Define $dVol_g = \sqrt{\det g_{ij}}\, dx^1 \wedge \cdots \wedge dx^n$, $dVol_g \in \Omega^n(M)$.

Well-defined: $dVol_g = f(x)\, dx^1 \wedge \cdots \wedge dx^n$. Need: $\tilde{f} = \det\left(\frac{\partial x}{\partial \tilde{x}}\right) f$ on overlaps

Know: $\tilde{g}_{ij} = \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial x^q}{\partial \tilde{x}^j} g_{pq}$

$\Rightarrow \det(\tilde{g}_{ij}) = \det\left(\frac{\partial x}{\partial \tilde{x}}\right) \det\left(\frac{\partial x}{\partial \tilde{x}}\right) \det g_{pq}$

Since M is orientable, can assume $\det\left(\frac{\partial x}{\partial \tilde{x}}\right) > 0$.

$$\therefore \sqrt{\det \tilde{g}_{ij}} = \det\left(\frac{\partial x}{\partial \tilde{x}}\right)\sqrt{\det g_{ij}}$$

$$\tilde{f} = \det\left(\frac{\partial x}{\partial \tilde{x}}\right) f \quad ✓$$

3. $dVol_g \in \Omega^n(M)$ is nowhere vanishing.

($\Leftarrow$) Suppose M admits nowhere vanishing $\omega \in \Omega^n(M)$.

1. Let $(\mathcal{U}, x^i)$ be coord chart. Write:

$$\omega = f(x)\, dx^1 \wedge \cdots \wedge dx^n \quad \text{and define:}$$

$$\eta_{\mathcal{U}} = \operatorname{sgn}(f(x)) \in \{\pm 1\}$$

2. Note the relation: on $(\mathcal{U}, x) \cap (\tilde{\mathcal{U}}, \tilde{x})$:

$$\tilde{f} = \det\left(\frac{\partial x}{\partial \tilde{x}}\right) f \quad \Rightarrow \quad +1 \overset{(*)}{=} \eta_{\tilde{\mathcal{U}}} \operatorname{sgn}\left(\det \frac{\partial x}{\partial \tilde{x}}\right) \eta_{\mathcal{U}}.$$

3. Define new coords on $\mathcal{U}$: old coords x^i
new coords y^i

$$\begin{cases} y^1 = \eta_{\mathcal{U}}\, x^1 \\ y^2 = x^2 \\ \vdots \\ y^n = x^n \end{cases}$$

4. Check:

$$\det \frac{\partial y}{\partial \tilde{y}} = \eta_{\mathcal{U}}\, \eta_{\tilde{\mathcal{U}}} \det \frac{\partial x}{\partial \tilde{x}} \overset{(*)}{>} 0. \qquad \square$$

<u>Def</u>: Suppose $F: M \to N$ is a local diffeo where both M and N are oriented. F is orientation preserving if
dF_p : {oriented basis} $\to$ {oriented basis} $\forall p \in M$.