16. Integration on Manifolds

Objects you can integrate $= \mu \in \Omega^n(M)$ over oriented mfd M with $\dim M = n$.

1) Let: $D \subseteq \mathcal{U} \subseteq M$, $(\mathcal{U}, x)$ coord chart, M oriented
: $\mu \in \Omega^n(M)$, cpt support in D.
: $\mu = f(x)\, dx^1 \wedge \cdots \wedge dx^n$

$$\int_D \mu := \int_{\mathbb{R}^n} f(x)\, dx^1 \cdots dx^n$$

Well-defined: if $D \subseteq \tilde{\mathcal{U}}$, $(\tilde{\mathcal{U}}, \tilde{x})$ another chart,
$\alpha = \tilde{f}(\tilde{x})\, d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n$, then

$$\int_{\mathbb{R}^n} \tilde{f}(\tilde{x})\, d\tilde{x}^1 \cdots d\tilde{x}^n = \int_{\mathbb{R}^n} \tilde{f}(x) \det\left(\frac{\partial \tilde{x}}{\partial x}\right) dx^1 \cdots dx^n$$

change of var formula

$$= \int_{\mathbb{R}^n} f(x)\, dx^1 \cdots dx^n \checkmark$$

$$f = \left(\det \frac{\partial \tilde{x}}{\partial x}\right) \tilde{f}$$

2) Let: M oriented, $\dim M = n$
: $\mu \in \Omega^n(M)$ with cpt support

$$\int_M \mu := \sum_\alpha \int_M \rho_\alpha \mu$$

where $\{\rho_\alpha\}$ is a partition of unity subordinate to $\{\mathcal{U}_\alpha\}$:

a) $\rho_\alpha : \mathcal{U}_\alpha \to \mathbb{R}_{\geq 0}$, $\operatorname{supp}(\rho_\alpha) \subset\subset \mathcal{U}_\alpha$
b) $\sum_\alpha \rho_\alpha = 1$

Well-defined: if take $\{\tilde{\rho}_\alpha\}$ alternate partition of unity,

$$\sum_\alpha \int_M \tilde{\rho}_\alpha \mu = \sum_{\alpha,\beta} \int_M \tilde{\rho}_\alpha \rho_\beta \mu = \sum_\beta \int_M \rho_\beta \mu \checkmark$$

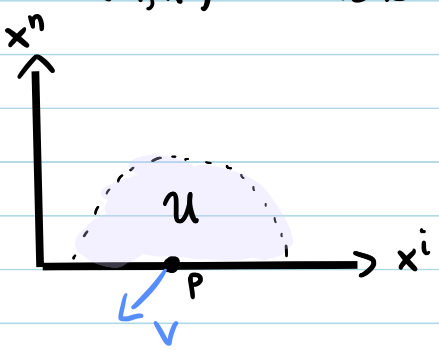
ex) $\displaystyle\int_{S^1} d\theta = \int_{S^1\setminus\{pt\}} d\theta = \int_0^{2\pi} d\theta = 2\pi$

(Recall coords: θ on $\mathcal{U} = \{e^{i\theta} : 0 < \theta < 2\pi\}$
$\tilde{\theta}$ on $\tilde{\mathcal{U}} = \{e^{i\tilde{\theta}} : -\pi < \theta < \pi\}$
$\tilde{\theta} = \begin{cases} \theta \\ \theta - 2\pi \end{cases} \Rightarrow S^1 = \mathcal{U} \cup \tilde{\mathcal{U}}$ oriented)

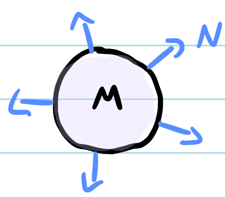
ex) $\displaystyle\int_{T^2} d\theta^1 \wedge d\theta^2 = (2\pi)(2\pi),\qquad T^2 = S^1 \times S^1$

Orientation of the boundary:

Def: Let M be mfd with boundary ∂M. Let $p \in \partial M$.
$V \in T_pM$ is outward-pointing if for some boundary coords $(\mathcal{U}, x^i)$ in a nbhd of p, V has negative x^n-component.



Lee Prop 15.33: Let (M,g) be Riemannian mfd with boundary.
$\exists!$ smooth outward-pointing $N \in \Gamma(TM|_{\partial M})$ s.t. $|N|_g \equiv 1$.



Orientation on ∂M: Equip M with a metric g.
Define $dVol_g \in \Omega^n(M)$.
Obtain outward-pointing N with $|N|_g = 1$.

Consider: $N \lrcorner\, dvol_g \in \Omega^{n-1}(\partial M)$. It is nowhere vanishing and defines an orientation on ∂M.

Stokes's Theorem: Let M be oriented mfd dim n.
Let $\omega \in \Omega^{n-1}(M)$ be compactly supported.

Then: $$\int_M d\omega = \int_{\partial M} \omega.$$

Note: $\int_{\partial M} \omega$ means: $\int_{\partial M} i^*\omega$, $i: \partial M \hookrightarrow M$.

Cor: Let M be a compact oriented mfd. Then:
$$\int_M d\omega = 0 \qquad \forall \omega \in \Omega^{n-1}(M).$$

Proof of Stokes's if $\partial M = \phi$:
$$\int_M d\omega = \int_M d\Big(\big(\sum_\alpha \rho_\alpha\big)\omega\Big)$$
ρ_α partion of unity subordinate to oriented coord cover $M = \bigcup_\alpha \mathcal{U}_\alpha$.
$$= \sum_\alpha \int_M d(\underbrace{\rho_\alpha \omega}_{\text{supp} \subseteq \mathcal{U}_\alpha})$$

Will show: $\int_{\mathbb{R}^n} d\eta = 0 \qquad \forall \eta \in \Omega_c^{n-1}(\mathbb{R}^n)$. e.g. $\eta = \rho_\alpha \omega$

Indeed: $$\int_{\mathbb{R}^n} d\eta = \int d\left(\frac{1}{(n-1)!}\eta_{i_1\cdots i_{n-1}} dx^{i_1}\wedge\cdots\wedge dx^{i_{n-1}}\right)$$
$$= \int \left(\partial_1 \eta_{23\cdots n} - \partial_2 \eta_{13\cdots n} + \partial_3 \eta_{124\cdots n} - \cdots\right) dx^1\wedge\cdots\wedge dx^n$$
$$= 0 - 0 + 0 - \cdots \quad \text{since e.g.}$$
$$\iint \left(\int_{-\infty}^{\infty} \frac{\partial}{\partial x^1}\eta_{23\cdots n}\, dx^1\right) dx^2\wedge\cdots\wedge dx^n$$
$$= \iint \left(\Big|_{-\infty}^{\infty} \eta_{23\cdots n}\right) dx^2\wedge\cdots\wedge dx^n = 0$$
since $\eta \equiv 0$ outside a compact set. $\square$

Proof of Stokes's Thm if $M = \mathbb{H}^n$:

Let $\omega \in \Omega^{n-1}(\mathbb{H})$ vanish on $|x| > R$.

Write: $\omega = \frac{1}{(n-1)!} \omega_{i_1 \cdots i_{n-1}} dx^{i_1} \wedge \cdots \wedge dx^{i_{n-1}}$

$$d\omega = \left(\partial_1 \omega_{23\cdots n} - \partial_2 \omega_{13\cdots n} + \partial_3 \omega_{12\cdots n} - \cdots \right) dx^1 \wedge \cdots \wedge dx^n.$$

$$\int_{\mathbb{H}^n} d\omega = \int_{\mathbb{H}^n} \underbrace{\partial_1 \omega_{23\cdots n} dx^1 \cdots dx^n}_{=0} - \int_{\mathbb{H}^n} \underbrace{\partial_2 \omega_{13\cdots n} dx^1 \cdots dx^n}_{=0} + \int_{\mathbb{H}^n} \underbrace{\partial_3 \omega_{12\cdots n} dx^1 \cdots dx^n}_{=0} - \cdots$$

$$= 0 - 0 + 0 - \cdots + (-1)^{n-1} \int_{\mathbb{H}^n} \partial_n \omega_{123\cdots n-1} dx^1 \cdots dx^n.$$

e.g. $$\int_{\mathbb{H}^n} \partial_1 \omega_{23\cdots n} dx^1 \cdots dx^n = \iint \cdots \left(\int_{-R}^{R} \partial_1 \omega_{23\cdots n} dx^1 \right) dx^2 \cdots dx^n$$

$$= \iint \cdots \left(\underbrace{\Big|_{x^1=-R}^{x^1=R} \omega_{23\cdots n}}_{=0} \right) dx^2 \cdots dx^n$$

$=0$ since $\omega \equiv 0$ on $|x| > R$

$$= 0.$$

$$\Rightarrow \int_{\mathbb{H}} d\omega = (-1)^{n-1} \int_{\mathbb{H}^n} \partial_n \omega_{12\cdots n-1} dx^1 \cdots dx^n$$

$$= (-1)^{n-1} \iint \cdots \left(\int_{x^n=0}^{x^n=R} \partial_n \omega_{12\cdots n-1} dx^n \right) dx^1 \cdots dx^{n-1}$$

$$= (-1)^{n-1} \iint \left(\Big|_{x^n=0}^{x^n=R} \omega_{12\cdots n-1} \right) dx^1 \cdots dx^n$$

$$= (-1)^n \iint \omega_{12\cdots n-1}(x^1, \ldots, x^{n-1}, 0)\, dx^1 \cdots dx^{n-1}.$$

$$= \pm \int_{\partial \mathbb{H}} i^* \omega$$ since $\partial \mathbb{H} = \{x^n = 0\}$

$i : \partial \mathbb{H} \to \mathbb{H}$, $i(x^1, \ldots, x^n) = (x^1, \ldots, x^{n-1}, 0)$

$$i^*\omega = i^* \left(\frac{1}{(n-1)!} \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right)$$

$$= \omega_{12\cdots n-1}(x^1, \ldots, x^{n-1}, 0)\, dx^1 \wedge \cdots \wedge dx^{n-1}.$$

What about $\pm$?

$$\int_{\partial \mathbb{H}^n} f(x^1,\dots,x^{n-1})\, dx^1\wedge\dots\wedge dx^{n-1} = \begin{cases} +\int_{\partial \mathbb{H}^n} f(x^1,\dots,x^{n-1})\, dx^1\dots dx^{n-1} & \text{if } \{\partial_1,\dots,\partial_{n-1}\} \text{ pos oriented} \\ -\int_{\partial \mathbb{H}^n} f(x^1,\dots,x^{n-1})\, dx^1\dots dx^{n-1} & \text{if } \{\partial_1,\dots,\partial_{n-1}\} \text{ neg oriented} \end{cases}$$

Boundary orientation: if $\{\partial_1,\dots,\partial_n\}$ pos oriented on $\mathbb{H}^n$, then

$\{\partial_1,\dots,\partial_{n-1}\}$ pos oriented on $\partial\mathbb{H}^n$ if: $\{\underset{\substack{\uparrow \\ \text{outward-pointing}}}{-\partial_n}, \partial_1,\dots,\partial_{n-1}\}$ pos oriented.

$$\begin{aligned} -\partial_n &= 0\,\partial_1 + \dots + 0\,\partial_{n-1} - 1\,\partial_n \\ \partial_1 &= 1\,\partial_1 + 0\,\partial_2 + \dots + 0\,\partial_n \\ &\vdots \\ \partial_{n-1} &= 0\,\partial_1 + \dots + 1\,\partial_{n-1} + 0\,\partial_n \end{aligned}$$

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & & & \\ \vdots & 1 & \ddots & & \\ 0 & & & 1 & 0 \end{pmatrix} \qquad \text{e.g.} \quad \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad n=3$$

$$\det \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = -1. \quad \text{Neg oriented.}$$

Can check:

$\{\partial_1,\dots,\partial_{n-1}\}$ pos oriented if n even

neg oriented if n odd

$$\therefore \int_{\partial\mathbb{H}^n} \omega_{12\cdots n-1}(x^1,\dots,x^{n-1},0)\, dx^1\dots dx^{n-1}.$$

$$= (-1)^n \int_{\partial\mathbb{H}^n} \omega_{12\cdots n-1}(x^1,\dots,x^{n-1},0)\, dx^1\wedge\dots\wedge dx^{n-1}$$

Can only "erase the wedges and integrate" if $dx^1\wedge\dots\wedge dx^{n-1}$ is s.t. $\{\partial_1,\dots,\partial_{n-1}\}$ is positively oriented. If not, need to include $-$ sign.

$$\therefore \int_{\mathbb{H}^n} d\omega = \underset{\substack{\uparrow \\ (-1)^n(-1)^n}}{+} \int_{\partial\mathbb{H}^n} i^*\omega.$$

<u>Remains to prove</u>: General Stokes's on M with $\partial M \neq \emptyset$. [Lee Thm 16.11]

ex) Show $\omega = \frac{x}{x^2+y^2}\,dy - \frac{y}{x^2+y^2}\,dx$ on $\mathbb{R}^2 \setminus \{0\}$ satisfies:

1) $d\omega = 0$
2) Cannot write $\omega = df$ for $f \in C^\infty(\mathbb{R}^2\setminus\{0\})$.

1) Calculation omitted.

2) Consider: $i : S^1 \hookrightarrow \mathbb{R}^2$
$i(e^{i\theta}) = (\cos\theta, \sin\theta)$

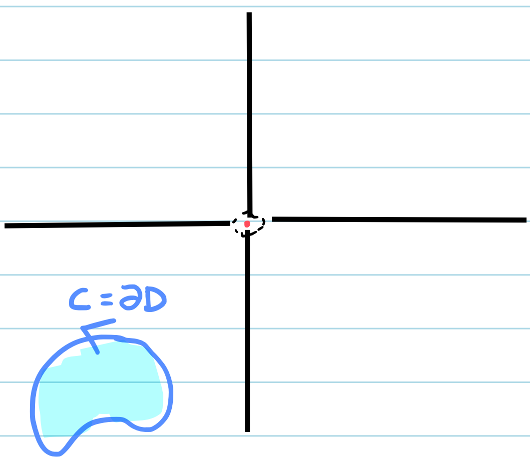
$$\begin{aligned} i^*\omega &= \cos\theta \, d\sin\theta - \sin\theta \, d\cos\theta \\ &= \cos^2\theta \, d\theta + \sin^2\theta \, d\theta \\ &= d\theta \end{aligned}$$

$$\int_{S^1} \omega := \int_{S^1} i^*\omega = \int_0^{2\pi} d\theta = 2\pi.$$

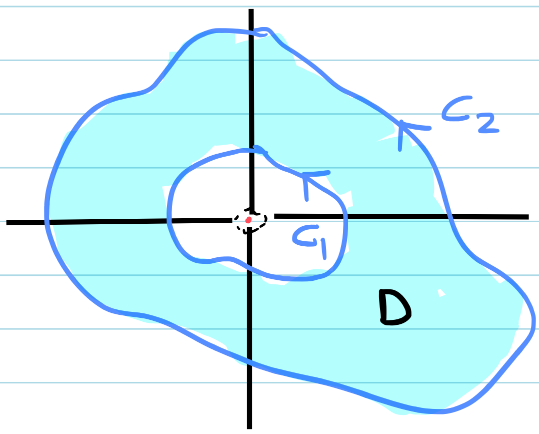
common notation: omit the i^*

If $\omega = df$, then $\int_{S^1}\omega := \int_{S^1} i^* df = \int_{S^1} d(f\circ i) = 0$ by Stokes's Thm.

ex) Let $\omega \in \Omega^1(\mathbb{R}^2\setminus\{0\})$ be s.t. $d\omega = 0$.



$$\int_C \omega = \int_{\partial D}\omega = \int_D d\omega = 0.$$

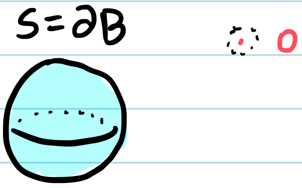


$\partial D = C_2 - C_1$

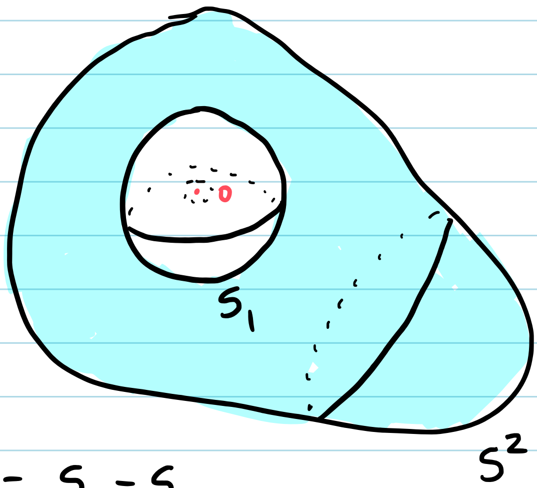
$$\int_{\partial D}\omega = 0 \Rightarrow \int_{C_1}\omega = \int_{C_2}\omega$$

ex) Let $\omega \in \Omega^2(\mathbb{R}^2 \setminus \{0\})$ be s.t. $d\omega = 0$.

$S = \partial B$



$$\int_S \omega = \int_{\partial B} \omega = \int_B d\omega = 0$$



$$\int_{S_1} \omega = \int_{S_2} \omega$$

$\partial B = S_2 - S_1$