17. De Rham Cohomology

$$H^k(M) = \frac{\operatorname{Ker}\{d : \Omega^k(M) \to \Omega^{k+1}(M)\}}{\operatorname{Im}\{d : \Omega^{k-1}(M) \to \Omega^k(M)\}}$$

k^{th} de Rham cohomology group

ex) $H^0(M) = \{ f \in C^\infty(M) : df = 0 \}$

Exercise: Let M be a connected mfd. Then $H^0(M) = \{\text{constant functions}\} \cong \mathbb{R}$.

Let $p \in M$. Show $\{f(x) = f(p)\} \subseteq M$ is open and closed.

ex) $\omega = \dfrac{x}{x^2+y^2}\,dx - \dfrac{y}{x^2+y^2}\,dy$. $\omega \in \Omega^1(M)$, $M = \mathbb{R}^2 \setminus \{0\}$

Showed earlier: $d\omega = 0$ but $\omega \neq df$ for any f.

$\therefore [\omega] \neq [0] \in H^1(M) \Rightarrow H^1(\mathbb{R}^2 \setminus \{0\}) \neq 0$.

ex) Consider $\omega \in \Omega^1(\mathbb{R}^2)$ with $d\omega = 0$.

$\omega = \omega_1 dx^1 + \omega_2 dx^2$.

Check: $\vec{F} = \begin{pmatrix} \omega_1(x,y) \\ \omega_2(x,y) \end{pmatrix}$ is conservative vector field.

2D curl $\vec{F} = (\partial_2 \omega_1 - \partial_1 \omega_2) = 0$ since $d\omega = 0$.

$\therefore \vec{F} = \nabla \varphi$ by classic Thm in vector calc.

$\therefore \begin{array}{l} \omega_1 = \partial_1 \varphi \\ \omega_2 = \partial_2 \varphi \end{array} \Rightarrow \omega = d\varphi \Rightarrow [\omega] = [0]$.

$\therefore H^1(\mathbb{R}^2) = 0$.

Similarly: $H^1(\mathbb{R}^n) = 0 \quad \forall n \geq 2$.

Homotopy:

Def: $\phi, \psi : M \to N$ are homotopic ($\phi \simeq \psi$) if
$\exists$ smooth $F: M \times [0,1] \to N$ s.t.
$F(x,0) = \phi$
$F(x,1) = \psi$.

Def: M and N are homotopic ($M \simeq N$) if
$\exists$ smooth maps $f: M \to N$ s.t. $g \circ f \simeq id$.
$g: N \to M$ $\quad f \circ g \simeq id$

ex) $\mathbb{R}^n \simeq \{pt\}$. $\quad f: \mathbb{R}^n \to \{0\}$
$g: \{0\} \hookrightarrow \mathbb{R}^n$

$g \circ f \simeq id$ via $F(x,t) = tx$

Deformation retract: Suppose $i: A \hookrightarrow M$ inclusion and $r: M \to A$ a map s.t. $r \circ i = id_A$. If $\exists F: M \times [0,1] \to M$ s.t. $F(x,0) = x$, $F(x,1) = i \circ r(x)$, then A is a deformation retract of M.

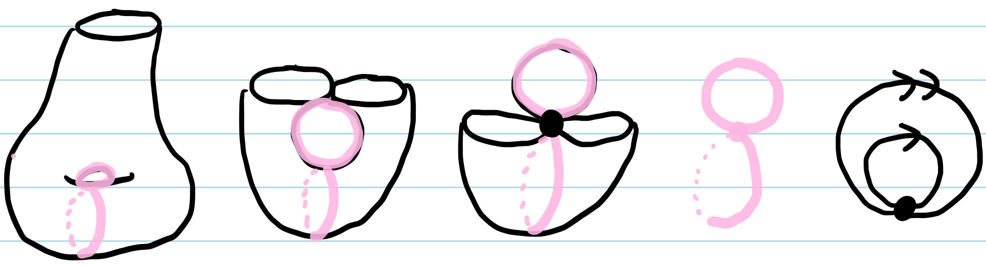
ex) $\mathbb{R}^2 \setminus \{0\} \simeq S^1$ via $r: \mathbb{R}^2 \setminus \{0\} \to S^1$, $r(x) = x/|x|$.
$F(x,t) = (1-t)x + t\, x/|x|$.

ex) $\mathbb{CP}^n \setminus \{[0,\dots,0,1]\} \simeq \mathbb{CP}^{n-1}$ via $F([z_0,\dots,z_n],t) = [z_0,\dots,z_{n-1},(1-t)z_n]$.
$\mathbb{CP}^{n-1} = \{[z_0,\dots,z_{n-1},0] \in \mathbb{CP}^n\}$

ex) $E \to M$ vector bundle. Then $E \simeq M$ retracts to zero section.

ex) 

$\mathbb{R}^2 \setminus \{0\}$, cylinder, Möbius strip all retract to S^1.

ex) 

retract to bouquet of two circles

Prop: Let $\omega \in \Omega^k(M \times I)$ be s.t. $d\omega = 0$.
Then $[i_1^*\omega] = [i_0^*\omega] \in H^k(M)$, where: $i_t : M \to M \times I$
$p \mapsto (p,t)$.

ex) $\omega = (1+x)dt + t\,dx$, $\omega \in \Omega^1(M \times \mathbb{R})$, $M = \mathbb{R}$.
$d\omega = dx \wedge dt + dt \wedge dx = 0$
$i_1^*\omega = dx$, $i_0^*\omega = 0$.
$i_1^*\omega - i_0^*\omega = d(\text{function on } M)$

Proof: Let $\Theta_t : M \times I \to M \times I$ flow of $\frac{\partial}{\partial t}$.
$\Theta_t(p,s) = (p, s+t)$

$$i_1^*\omega - i_0^*\omega = \int_0^1 \frac{d}{dt} i_t^*\omega \, dt$$

Note: $$\frac{d}{dt} i_t^*\omega = \frac{d}{dt} i_0^* \Theta_t^* \omega = i_0^* \frac{d}{dt} \Theta_t^* \omega$$
$$= \underbrace{i_0^* \Theta_t^*}_{i_t^*} L_{\frac{\partial}{\partial t}} \omega$$ Lie derivative

$$\Rightarrow i_1^*\omega - i_0^*\omega = \int_0^1 i_t^* L_{\frac{\partial}{\partial t}} \omega \, dt$$
$$= \int_0^1 i_t^* \Big(d\, \tfrac{\partial}{\partial t} \lrcorner\, \omega + \tfrac{\partial}{\partial t} \lrcorner\, \underbrace{d\omega}_{=0} \Big) dt$$ Cartan's formula
$$= d \int_0^1 i_t^* \left(\tfrac{\partial}{\partial t} \lrcorner\, \omega \right) dt$$
$$= d(\text{something})$$

Thm: Let $F : M \to N$, $G : M \to N$ be homotopic smooth maps. Let $\omega \in \Omega^k(N)$ with $d\omega = 0$.

Then: $[F^*\omega] = [G^*\omega]$.

Pf: Let $H : M \times I \to M$ be a homotopy from F to G.
$[F^*\omega] = [(H \circ i_0)^*\omega] = [i_0^* H^*\omega] = [i_1^* H^*\omega] = [(H \circ i_1)^*\omega] = [G^*\omega]$. $\square$

Note: Given $\varphi: M \to N$, can define $\qquad d\varphi^*\eta = \varphi^* d\eta = 0$

$\varphi^*: H^k(N) \to H^k(M)$ by: $\varphi^*[\eta] = [\varphi^*\eta]$.

Thm: $M \simeq N \Rightarrow H^k(M) \cong H^k(N)$.

Pf: Take $f: M \to N$ with $g \circ f \simeq id$
$g: N \to M$ $\qquad f \circ g \simeq id$.

$\Rightarrow (f \circ g)^*[\eta] = [\eta] \in H^k(N)$ $\qquad$ Note: $(g \circ f)^* = f^* g^*$
$(g \circ f)^*[\alpha] = [\alpha] \in H^k(M)$

$\Rightarrow g^* f^* [\eta] = [\eta]$ $\qquad$ inverses
$f^* g^* [\alpha] = [\alpha]$

$\Rightarrow f^*: H^k(N) \to H^k(M)$ is isomorphism. $\square$

Cor: $H^k(\mathbb{R}^n) = 0 \quad \forall k \geq 1$.

Cor: If $\omega \in \Omega^k(\mathbb{R}^n)$, $k \geq 1$, and $d\omega = 0$, then $\omega = d\alpha$ for some $\alpha \in \Omega^{k-1}(\mathbb{R}^n)$.

Cor: Let M be a smooth manifold.
$\forall p \in M$, $\exists$ nbhd $\mathcal{U}$ on which every closed form is exact.

Closed form: $\omega \in \Omega^k(M)$ with $d\omega = 0$.

Exact form: $\omega \in \Omega^k(M)$ with $\omega = d\alpha$ for some $\alpha \in \Omega^{k-1}(M)$.

18. Mayer-Vietoris Sequence

Let $C = \bigoplus_{k \in \mathbb{Z}} C^k$ be a direct sum of vector spaces.

C is a differential complex if $\exists$ homomorphisms

$$\cdots \to C^{k-1} \xrightarrow{d} C^k \xrightarrow{d} C^{k+1} \to \cdots \quad \text{s.t.} \quad d^2 = 0.$$

<u>Cohomology of C:</u> $\quad H^k(C) = \dfrac{\operatorname{Ker}\{d : C^k \to C^{k+1}\}}{\operatorname{Im}\{d : C^{k-1} \to C^k\}}$

<u>Chain map:</u> $f : A \to B$ between differential complexes s.t. $f d_A = d_B f$.

<u>Exact sequence:</u> Vector spaces V_k with homomorphisms

$$\cdots \to V_{k-1} \xrightarrow{f_{k-1}} V_k \xrightarrow{f_k} V_{k+1} \xrightarrow{f_{k+1}} \cdots \quad \text{s.t.} \quad \operatorname{Ker} f_i = \operatorname{Im} f_{i-1} \quad \forall i.$$

<u>Snake Lemma:</u> Given a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

where: A, B, C differential complexes
f, g chain maps,

then $\exists$ long exact sequence in cohomology

$$\cdots \to H^k(A) \xrightarrow{f} H^k(B) \xrightarrow{g} H^k(C) \xrightarrow{\delta} H^{k+1}(A) \xrightarrow{f} H^{k+1}(B) \xrightarrow{g} H^{k+1}(C) \xrightarrow{\delta} \cdots$$

where $\delta[c] = [f^{-1} d \, g^{-1}(c)]$.

<u>Proof:</u> Let $c \in C^k$ with $dc = 0$. $\delta[c] = [a]$, where:

$$\begin{array}{ccccccccc} & & a & & db & & & & \\ 0 & \to & A^{k+1} & \xrightarrow{f} & B^{k+1} & \xrightarrow{g} & C^{k+1} & \to & 0 \\ & & \uparrow d & & \uparrow d & & \uparrow d & & \\ 0 & \to & A^k & \xrightarrow{f} & B^k & \xrightarrow{g} & C^k & \to & 0 \\ & & & & b & & c & & \end{array}$$

$g(db) = d\,g(b) = dc = 0.$

$f\,da = d f a = dd\,b = 0$
$\Rightarrow da = 0.$

Must check: $[a] \in H^{k+1}(A)$ is indep of: $c \in [c]$

: choice of b

: choice of a

: long exact sequence is exact.

Omitted.

$\square$

Back to de Rham cohomology

Suppose $M = \mathcal{U} \cup V$ union of open sets. Consider:

$$0 \to \Omega^k(M) \to \Omega^k(\mathcal{U}) \oplus \Omega^k(V) \to \Omega^k(\mathcal{U} \cap V) \to 0$$

$$\omega \qquad \gamma \qquad -\omega + \gamma$$

1) All maps are chain maps
2) Sequence is exact.

e.g. if $\omega \in \Omega^k(\mathcal{U} \cap V)$, then $\omega = -(-\rho_V \omega) + \rho_{\mathcal{U}} \omega$

$$\uparrow$$

$(-\rho_V \omega, \rho_{\mathcal{U}} \omega) \in \Omega^k(\mathcal{U}) \oplus \Omega^k(V)$

where: $\rho_{\mathcal{U}} + \rho_V = 1$

$\operatorname{supp} \rho_{\mathcal{U}} \subseteq \mathcal{U}$, $\operatorname{supp} \rho_V \subseteq V$.

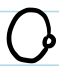
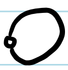
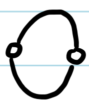
<u>NB</u>: $\rho_V \omega \in \Omega^k(\underline{\underline{\mathcal{U}}})$

$\mathcal{U}$ V

By the Snake Lemma, $\exists$ long exact seq in cohomology:

$$\cdots \to H^k(M) \to H^k(\mathcal{U}) \oplus H^k(V) \to H^k(\mathcal{U} \cap V) \to H^{k+1}(M) \to \cdots$$

called the Mayer-Vietoris sequence.

ex) $S^1 =$  $\cup$  , $\mathcal{U} \cap V =$  $\simeq \{pt\} \sqcup \{pt\}$

$\mathcal{U}$ V

$\mathcal{U} \simeq \{pt\}$, $V \simeq \{pt\}$

$$0 \to H^0(S^1) \to H^0(\mathcal{U}) \oplus H^0(V) \to H^0(\mathcal{U} \cap V) \to$$
$$\to H^1(S^1) \to H^1(\mathcal{U}) \oplus H^1(V) \to H^1(\mathcal{U} \cap V) \to 0$$

$$0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}^2 \to H^1(S^1) \to 0$$

dimker = 0, dimIm = 1 | dimker = 1, dimIm = 1 | dimker = 1, dimIm = 1 | dimker = 1, dimIm = 0

Recall: $T: V \to W$ linear map,

$\dim V = \dim \operatorname{Ker} T + \dim \operatorname{Im} T$.

$\Rightarrow \dim H^1(S^1) = 1.$

Consider $d\theta \in \Omega^1(S^1)$. Note $\int_{S^1} d\theta = 2\pi$.

$\Rightarrow [d\theta] \neq 0 \in H^1(S^1).$

$\left(\text{if } [\alpha]=0, \text{ then } \int_{S^1} \alpha = \int_{S^1} df = 0 \right)$

$\Rightarrow [d\theta]$ is a generator of $H^1(S^1)$.

Note: Poincaré Duality [Lee Problem 18-7]

$H^k(M) \cong (H^{n-k}(M))^*$ for M compact oriented.

$\Rightarrow \dim H^k(M) = \dim H^{n-k}(M)$

$\Rightarrow \dim H^n(M) = 1$, $\dim M = n$, M compact connected oriented.

We will use this freely in the following examples.

Def: $b_i = \dim H^i(M)$, i^{th} Betti number

Def: $\chi(M) = \sum_{i=0}^{n} (-1)^i b_i$.

ex) M^2 compact connected oriented surface.

$\Rightarrow \chi(M) = 1 - b_1 + 1 = 2 - \dim H^1(M^2)$

Prop: $M = \mathcal{U} \cup V$ union of open sets.

$\Rightarrow \chi(M) = \chi(\mathcal{U}) + \chi(V) - \chi(\mathcal{U} \cap V).$

Pf: Given an exact seq of vector spaces

$0 \to V_1 \to V_2 \to \cdots \to V_k \to 0,$

then $0 = \sum_{i=1}^{k} (-1)^k \dim V_i$.

(use $\dim V = \dim \ker + \dim \mathrm{Im}$)

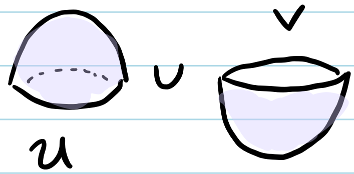
Apply this to Mayer-Vietoris:

$$0 \to H^0(M) \to H^0(\mathcal{U}) \oplus H^0(V) \to H^0(\mathcal{U} \cap V) \to H^1(M) \to \cdots \to 0.$$

$$\Rightarrow 0 = \sum_{i=0}^{n} (-1)^i \dim H^i(M) - \sum (-1)^i \dim H^i(\mathcal{U}) + \sum (-1)^i \dim H^i(\mathcal{U} \cap V)$$
$$- \sum (-1)^i \dim H^i(V)$$

$$\Rightarrow 0 = \chi(M) - \chi(\mathcal{U}) - \chi(V) + \chi(\mathcal{U} \cap V).$$

$\square$

ex) $S^2 =$  ,

$\mathcal{U}$ V

$\mathcal{U} \simeq \{pt\}$

$V \simeq \{pt\}$

$\mathcal{U} \cap V =$  $= S^1$

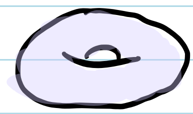
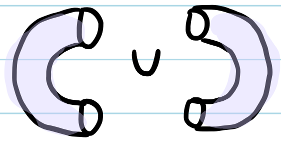
$$\chi(S^2) = \chi(\mathcal{U}) + \chi(V) - \chi(\mathcal{U} \cap V)$$
$$= 1 + 1 - 0$$

$0 = \chi(S^1) = b_0 - b_1$

$\Rightarrow 2 - \dim H^1 = 2$

$\Rightarrow \dim H^1(S^2) = 0.$

$$\Rightarrow \begin{cases} H^0(S^2) = \mathbb{R} \\ H^1(S^2) = 0 \\ H^2(S^2) = \mathbb{R} \end{cases}$$

ex) $T^2 =$  $=$  ,

$\mathcal{U}$ V

$\mathcal{U} \simeq \bigcirc = S^1$

$V \simeq \bigcirc = S^1$

$\mathcal{U} \cap V \simeq \bigcirc \sqcup \bigcirc = S^1 \sqcup S^1$

$$\chi(T^2) = \chi(\mathcal{U}) + \chi(V) - \chi(\mathcal{U} \cap V)$$
$$= 0 + 0 - 0$$

$\Rightarrow 2 - b_1 = 0 \Rightarrow b_1 = 2$

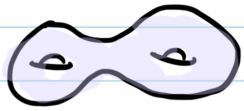
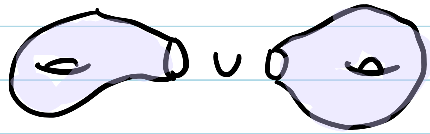
$$\Rightarrow \begin{cases} H^0(T^2) = \mathbb{R} \\ H^1(T^2) = \mathbb{R}^2 \\ H^2(T^2) = \mathbb{R} \end{cases}$$

Generators of $H^1(T^2)$? Write $T^2 = \underset{d\theta^1}{S^1} \times \underset{d\theta^2}{S^1}$

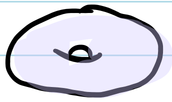
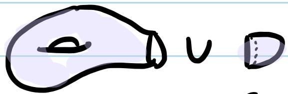
$H^1(T^2) = \text{span}\{[d\theta^1], [d\theta^2]\}.$

These cannot be linearly dependent: if $d\theta^1 = d\theta^2 + df$, $f \in C^\infty(T^2)$,

$$2\pi = \int_{S^1 \times \{pt\}} d\theta^1 = \int_{S^1 \times \{pt\}} d\theta^2 + \int_{S^1 \times \{pt\}} df = 0 + 0. \quad \Rightarrow\Leftarrow$$

ex) Σ_g, genus $g=2$ =  = 

$\mathcal{U}$ $\quad$ V

 = 

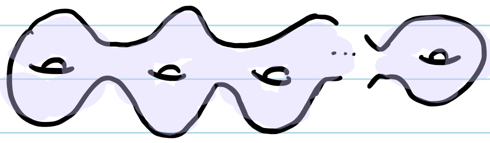
T^2 $\quad$ $\mathcal{U}$ $\quad$ Cap $\quad\quad$ Cap $\simeq \{pt\}$

$$\chi(T^2) = \chi(\mathcal{U}) + \chi(\{pt\}) - \chi(S^1)$$
$$0 = \chi(\mathcal{U}) + 1 \quad \Rightarrow \chi(\mathcal{U}) = -1.$$

$$\chi(\Sigma_g) = \chi(\mathcal{U}) + \chi(V) - \chi(\mathcal{U} \cap V)$$
$$= -1 - 1 - 0 \quad\quad = \chi(S^1)$$

$\Rightarrow \dim H^1(\Sigma_g) = 4.$

General Pattern: $\chi(\Sigma_g) = 2 - 2g$, genus g surface.

 $= T^2 \# T^2 \# \cdots \# T^2$

Exercise: $H^k(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise} \end{cases}$

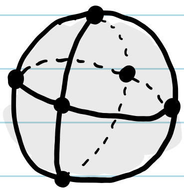
Exercise: $H^k(\mathbb{CP}^n) = \begin{cases} \mathbb{R} & k = 0, 2, 4, \dots, 2n \quad \text{even} \\ 0 & \text{otherwise} \end{cases}$

One more remark on $\chi(M^2)$: Given a triangulation of a compact surface M^2, then:

$$\chi(M^2) = V - E + F.$$

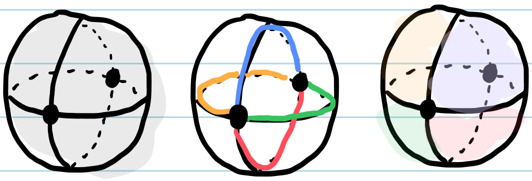
Triangulation of M^2: collection of polygons, each contained in a single coordinate chart, s.t.

1) Every $p \in M$ inside at least 1 polygon
2) Two polygons are disjoint or their intersection is an edge or common vertex
3) Each edge is the edge of exactly 2 polygons.

ex) 

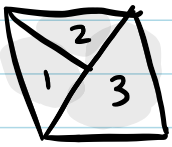
use octants as triangles

$$\chi(S^2) = 6 - 12 + 8 = 2$$

ex) 

use quarters as 2-gons

$$\chi(S^2) = 2 - 4 + 4 = 2$$

ex) 

not valid

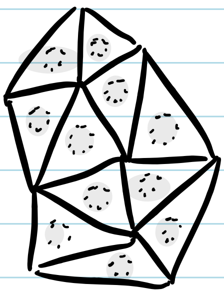
ex) use halves as 1-gons

$$\chi = 1 - 1 + 2$$

Proof of $\chi = V - E + F$:

Write $M = \mathcal{U} \cup M \setminus \{p_1, \dots, p_F\}$

where: take one point p_i in each polygon. (# polygons = F)
: $\mathcal{U}$ = disjoint union of balls centred at p_i.



$$\mathcal{U} \simeq \coprod_F \{pt\}$$

$$\mathcal{U} \cap (M \setminus \{p_1, \dots, p_F\}) \simeq \coprod_F S^1$$

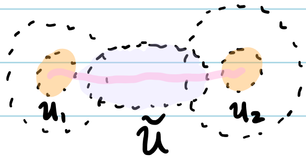
$$\chi(M) = \chi(\mathcal{U}) + \chi(M \setminus \{p_1, \dots, p_F\}) - \chi(\mathcal{U} \cap (M \setminus \{p_1, \dots, p_F\}))$$

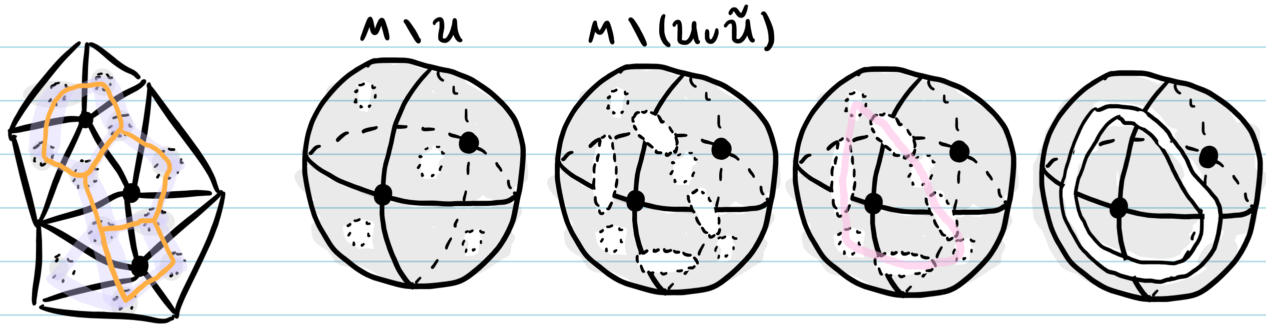
$$\chi(M) = F + \chi(M \setminus \{p_1, \dots, p_F\}). \qquad (*)$$

Next: $M \setminus \{p_1, \dots, p_F\} = \tilde{\mathcal{U}} \cup M \setminus \{\text{arcs}\}$

where: $\tilde{\mathcal{U}}$ = disjoint union of contractible opens, one for each edge, linking balls of old $\mathcal{U}$ with 2x radius.

arcs = paths in $2\mathcal{U} \cup \tilde{\mathcal{U}}$ joining the p_i.





$\tilde{U} = \bigsqcup_{E}$

$\tilde{U} \cap (M \setminus \{\text{arcs}\}) = \bigsqcup_{E}$

$M \setminus \{\text{arcs}\} \simeq \bigsqcup_{V}$

$\simeq \bigsqcup_{2E}$

$\Rightarrow \chi(M \setminus \{p_1, \dots, p_F\}) = E + V - 2E$

$\Rightarrow \chi(M) = F - E + V.$

(*) $\square$