

18. Mayer-Vietoris Sequence

Let $C = \bigoplus_{k \in \mathbb{Z}} C^k$ be a direct sum of vector spaces.

C is a differential complex if $\exists$ homomorphisms

$$\dots \rightarrow C^{k-1} \xrightarrow{d} C^k \xrightarrow{d} C^{k+1} \rightarrow \dots \quad \text{s.t. } d^2 = 0.$$

Cohomology of C : $H^k(C) = \frac{\text{Ker } \{d: C^k \rightarrow C^{k+1}\}}{\text{Im } \{d: C^{k-1} \rightarrow C^k\}}$

Chain map: $f: A \rightarrow B$ between differential complexes
s.t. $f d_A = d_B f$.

Exact sequence: Vector spaces V_k with homomorphisms

$$\dots \rightarrow V_{k-1} \xrightarrow{f_{k-1}} V_k \xrightarrow{f_k} V_{k+1} \rightarrow \dots \quad \text{s.t. } \text{Ker } f_i = \text{Im } f_{i-1} \quad \forall i.$$

Snake Lemma: Given a short exact sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

where: A, B, C differential complexes
 f, g chain maps,

then $\exists$ long exact sequence in cohomology

$$\dots \rightarrow H^k(A) \xrightarrow{f} H^k(B) \xrightarrow{g} H^k(C) \xrightarrow{\delta} H^{k+1}(A) \xrightarrow{f} H^{k+1}(B) \xrightarrow{g} H^{k+1}(C) \xrightarrow{\delta} \dots$$

where $\delta[c] = [f^{-1} d g^{-1}(c)]$.

Proof: Let $c \in C^k$ with $dc = 0$. $\delta[c] = [a]$, where:

$$\begin{array}{ccccccc} 0 & \rightarrow & A^{k+1} & \xrightarrow{f} & B^{k+1} & \xrightarrow{g} & C^{k+1} \rightarrow 0 \\ & & \uparrow d & & \uparrow d & & \uparrow d \\ 0 & \rightarrow & A^k & \xrightarrow{f} & B^k & \xrightarrow{g} & C^k \rightarrow 0 \end{array} \quad \begin{array}{l} \text{a} \quad \text{db} \\ g(db) = dg(b) = dc = 0. \\ f da = d f a = dd b = 0 \\ \Rightarrow da = 0. \end{array}$$

Must check: $[a] \in H^{k+1}(A)$ is indep of: $c \in [c]$, check long exact
: choice of b seq is exact.
Omitted. : choice of a

□

Back to de Rham cohomology

Suppose $M = U \cup V$ union of open sets. Consider:

$$0 \rightarrow \Omega^k(M) \xrightarrow{\text{restrict}} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{\text{difference}} \Omega^k(U \cap V) \rightarrow 0$$

$\omega \quad \gamma \quad -\omega + \gamma$

1) All maps are chain maps

2) Sequence is exact.

e.g. if $\omega \in \Omega^k(U \cap V)$, then $\omega = -(-\rho_V \omega) + \rho_U \omega$

where: $\rho_U + \rho_V = 1$
 $\text{supp } \rho_U \subseteq U, \text{ supp } \rho_V \subseteq V$.
 $(-\rho_V \omega, \rho_U \omega) \in \Omega^k(U) \oplus \Omega^k(V)$
NB: $\rho_V \omega \in \Omega^k(U)$

By the Snake Lemma, $\exists$ long exact seq in cohomology:



$$(*) \quad \dots \rightarrow H^k(M) \xrightarrow{\text{restrict}} H^k(U) \oplus H^k(V) \xrightarrow{\text{difference}} H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(M) \rightarrow \dots$$

$[\omega] \quad [\gamma] \quad [-\omega + \gamma]$

$$\delta = "f^{-1} d g^{-1}" \Rightarrow \delta[\eta] = [d(\rho_U \eta)]. \quad \delta: H^k(U \cap V) \rightarrow H^{k+1}(M)$$

(*) = Mayer-Vietoris sequence.

ex) $S^1 = \bigcirc_U \cup \bigcirc_V$, $U \cap V = \bigcirc \cong \{pt\} \sqcup \{pt\}$
 $U \cong \{pt\}, V \cong \{pt\}$

$$0 \rightarrow H^0(S^1) \rightarrow H^0(U) \oplus H^0(V) \rightarrow H^0(U \cap V) \rightarrow$$

$$\rightarrow H^1(S^1) \rightarrow H^1(U) \oplus H^1(V) \rightarrow H^1(U \cap V) \rightarrow 0$$

$$0 \rightarrow \mathbb{R} \rightarrow \mathbb{R} \oplus \mathbb{R} \rightarrow \mathbb{R}^2 \rightarrow H^1(S^1) \rightarrow 0$$

$\dim \ker = 0 \quad \dim \ker = 1 \quad \dim \ker = 1 \quad \dim \ker = 1$
 $\dim \text{Im} = 1 \quad \dim \text{Im} = 1 \quad \dim \text{Im} = 1 \quad \dim \text{Im} = 0$

Recall: $T: V \rightarrow W$ linear map,
 $\dim V = \dim \ker T + \dim \text{Im } T$.

$$\Rightarrow \dim H^1(S') = 1.$$

Consider $d\theta \in \Omega^1(S')$. Note $\int_{S'} d\theta = 2\pi$.

$$\Rightarrow [d\theta] \neq 0 \in H^1(S').$$

$\Rightarrow [d\theta]$ is a generator of $H^1(S')$.

$$\text{if } [\alpha] = 0, \text{ then } \int_{S'} \alpha = \int_{S'} df = 0$$

Poincaré Duality [Lee Problem 18-7]

M^n compact + orientable. Then $\dim H^k(M) = \dim H^{n-k}(M)$, and

$$\text{PD}: H^k(M) \xrightarrow[\text{iso}]{\sim} H^{n-k}(M)^*$$

$$[\alpha] \mapsto \int_M \alpha \wedge \cdot$$

Will not prove Poincaré duality.

Def: $b_i = \dim H^i(M)$, i^{th} Betti number

Def: $\chi(M) = \sum_{i=0}^n (-1)^i b_i$.

ex) M^2 compact connected oriented surface. $\text{PD} \Rightarrow H^2(M) \cong H^0(M) \cong \mathbb{R}$.
 $\Rightarrow \chi(M) = 1 - b_1 + 1 = 2 - \dim H^1(M^2)$

Prop: $M = U \cup V$ union of open sets.

$$\Rightarrow \chi(M) = \chi(U) + \chi(V) - \chi(U \cap V).$$

Pf: Given an exact seq of vector spaces

$$0 \rightarrow V_1 \rightarrow V_2 \rightarrow \dots \rightarrow V_K \rightarrow 0,$$

$$\text{then } 0 = \sum_{i=1}^K (-1)^i \dim V_i.$$

(use $\dim V = \dim \ker + \dim \text{Im}$)

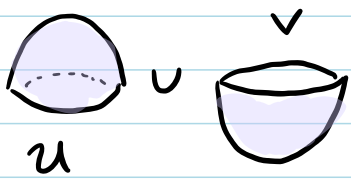
Apply this to Mayer-Vietoris:

$$0 \rightarrow H^0(M) \rightarrow H^0(U) \oplus H^0(V) \rightarrow H^0(U \cap V) \rightarrow H^1(M) \rightarrow \dots \rightarrow 0.$$

$$\Rightarrow 0 = \sum_{i=0}^n (-1)^i \dim H^i(M) - \sum (-1)^i \dim H^i(U) + \sum (-1)^i \dim H^i(U \cap V) - \sum (-1)^i \dim H^i(V)$$

$$\Rightarrow 0 = \chi(M) - \chi(U) - \chi(V) + \chi(U \cap V).$$

□

ex) $S^2 =$ , $U \cong \{\text{pt}\}$
 $V \cong \{\text{pt}\}$
 $U \cap V =$  $\cong$  $= S^1$

$$\chi(S^2) = \chi(U) + \chi(V) - \chi(U \cap V)$$

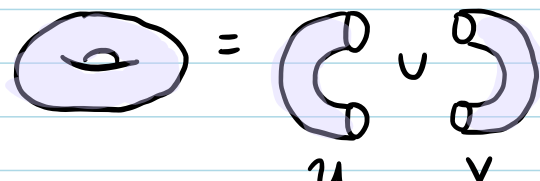
$$= 1 + 1 - 0$$

$$\chi(S^1) = b_0 - b_1$$

$$\Rightarrow 2 - \dim H^1 = 2$$

$$\Rightarrow \dim H^1(S^2) = 0.$$

$$\Rightarrow \begin{cases} H^0(S^2) = \mathbb{R} \\ H^1(S^2) = 0 \\ H^2(S^2) = \mathbb{R} \end{cases}$$

ex) $T^2 =$ , $U \cong O = S^1$
 $V \cong O = S^1$
 $U \cap V \cong O \sqcup O = S^1 \sqcup S^1$

$$\chi(T^2) = \chi(U) + \chi(V) - \chi(U \cap V)$$

$$= 0 + 0 - 0$$

$$\Rightarrow 2 - b_1 = 0 \Rightarrow b_1 = 2$$

$$\Rightarrow \begin{cases} H^0(T^2) = \mathbb{R} \\ H^1(T^2) = \mathbb{R}^2 \\ H^2(T^2) = \mathbb{R} \end{cases}$$

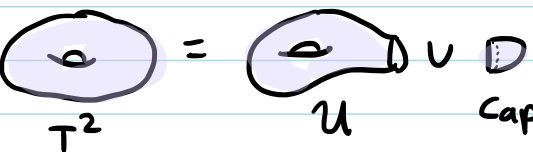
Generators of $H^1(T^2)$? Write $T^2 = S^1 \times S^1$
 $d\theta^1 \quad d\theta^2$

$$H^1(T^2) = \text{span}\{[d\theta^1], [d\theta^2]\}.$$

These cannot be linearly dependent: if $d\theta^1 = d\theta^2 + df$, $f \in C^\infty(T^2)$,

$$2\pi = \int_{S^1 \times \{\text{pt}\}} d\theta^1 = \int_{S^1 \times \{\text{pt}\}} d\theta^2 + \int_{S^1 \times \{\text{pt}\}} df = 0 + 0. \quad \Rightarrow \Leftarrow$$

ex) Σ_g , genus $g=2 =$  $= U \cup V$

 $= U \cup \text{Cap}$ $\text{Cap} \simeq \{\text{pt}\}$

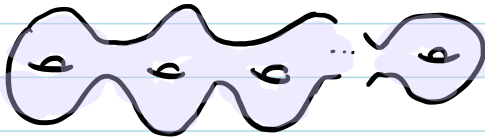
$$\chi(T^2) = \chi(U) + \chi(\{\text{pt}\}) - \chi(S')$$

$$0 = \chi(U) + 1 \Rightarrow \chi(U) = -1.$$

$$\begin{aligned} \chi(\Sigma_g) &= \chi(U) + \chi(V) - \chi(U \cap V) \\ &= -1 -1 -0 = \chi(S') \end{aligned}$$

$$\Rightarrow \dim H^1(\Sigma_g) = 4.$$

General Pattern: $\chi(\Sigma_g) = 2 - 2g$, genus g surface.

 $= T^2 \# T^2 \# \dots \# T^2$

Exercise: $H^k(S^n) = \begin{cases} \mathbb{R} & k=0, n \\ 0 & \text{otherwise} \end{cases}$

Exercise: $H^k(\mathbb{C}P^n) = \begin{cases} \mathbb{R} & k=0, 2, 4, \dots, 2n \text{ even} \\ 0 & \text{otherwise} \end{cases}$

One more remark on $\chi(M^2)$: Given a triangulation of a compact surface M^2 , then:

$$\chi(M^2) = V - E + F.$$

Triangulation of M^2 : collection of polygons, each contained in a single coordinate chart, s.t.

- 1) Every $p \in M$ inside at least 1 polygon
- 2) Two polygons are disjoint or their intersection is an edge or common vertex
- 3) Each edge is the edge of exactly 2 polygons.

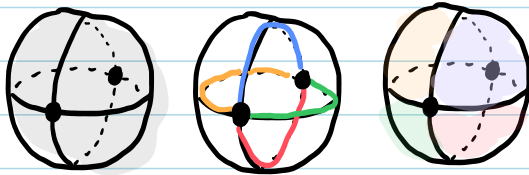
ex)



use octants as triangles

$$\chi(S^2) = 6 - 12 + 8 = 2$$

ex)



use quarters as 2-gons

$$\chi(S^2) = 2 - 4 + 4 = 2$$

ex)



not valid

ex) use halves as 1-gons

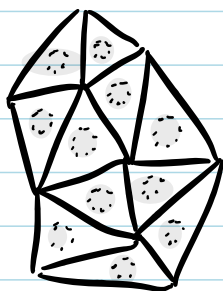


$$\chi = 1 - 1 + 2$$

Proof of $\chi = V - E + F$:

Write $M = \mathcal{U} \cup M \setminus \{p_1, \dots, p_F\}$

where: take one point p_i in each polygon. ($\# \text{polygons} = F$)
: $\mathcal{U}$ = disjoint union of balls centred at p_i .



$$\mathcal{U} \simeq \bigsqcup_F \{pt\}$$

$$\mathcal{U} \cap (M \setminus \{p_1, \dots, p_F\}) \simeq \bigsqcup_F S^1$$

$$\chi(M) = \chi(\mathcal{U}) + \chi(M \setminus \{p_1, \dots, p_F\}) - \chi(\mathcal{U} \cap (M \setminus \{p_1, \dots, p_F\}))$$

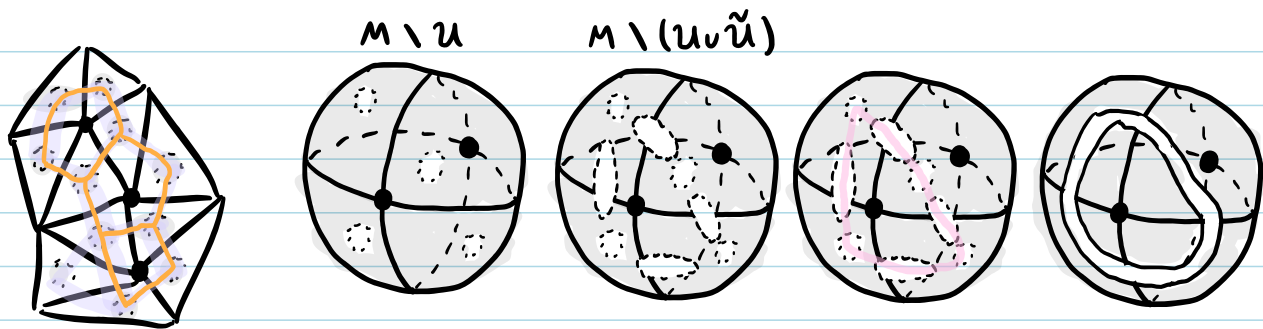
$$\chi(M) = F + \chi(M \setminus \{p_1, \dots, p_F\}). \quad (*)$$

Next: $M \setminus \{p_1, \dots, p_F\} = \tilde{\mathcal{U}} \cup M \setminus \{\text{arcs}\}$

where: $\tilde{\mathcal{U}}$ = disjoint union of contractible opens, one for each edge, linking balls of old $\mathcal{U}$ with $2\times$ radius.

arcs = paths in $2\mathcal{U} \cup \tilde{\mathcal{U}}$ joining the p_i .





$$\tilde{u} = \bigsqcup_E \text{ (disk with diagonal lines) }$$

$$\tilde{u} \cap (M \setminus \{\text{arcs}\}) = \bigsqcup_E \text{ (purple shaded disk) }$$

$$M \setminus \{\text{arcs}\} \simeq \bigsqcup_V \text{ (disk with diagonal lines) }$$

$$\simeq \bigsqcup_{2E} \text{ (disk with diagonal lines) }$$

$$\Rightarrow \chi(M \setminus \{p_1, \dots, p_F\}) = E + V - 2E$$

$$\Rightarrow \chi(M) = F - E + V.$$

(*)

