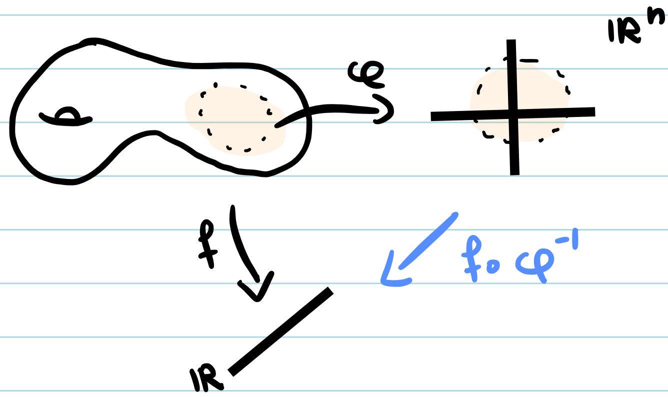
2. Smooth Maps

Def: M smooth mfd.
Say $f: M \to \mathbb{R}$ is smooth $(f \in C^\infty(M))$ if
$f \circ \varphi^{-1}$ is smooth $\forall (\mathcal{U}, \varphi) \in \mathcal{A}$.

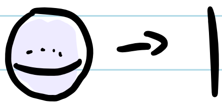


Notation: Over chart $(\mathcal{U}, \varphi)$, we often just write

$$f \overset{\text{loc}}{=} f(x^1, \dots, x^n) \quad \text{using coords } x^i$$

$$\text{instead of } f \circ \varphi^{-1} = f(x^1, \dots, x^n)$$

ex) height function $h: S^2 \to \mathbb{R}$, $h(x,y,z) = z$.



Over chart: $\mathcal{U} = \{x > 0\}$
$\varphi(\sqrt{1-u^2-v^2}, u, v) = (u, v)$

$$h(u,v) \overset{\text{loc}}{=} v$$

Over chart: $\tilde{\mathcal{U}} = \{z > 0\}$
$\tilde{\varphi}(\tilde{u}, \tilde{v}, \sqrt{1-\tilde{u}^2-\tilde{v}^2}) = (\tilde{u}, \tilde{v})$

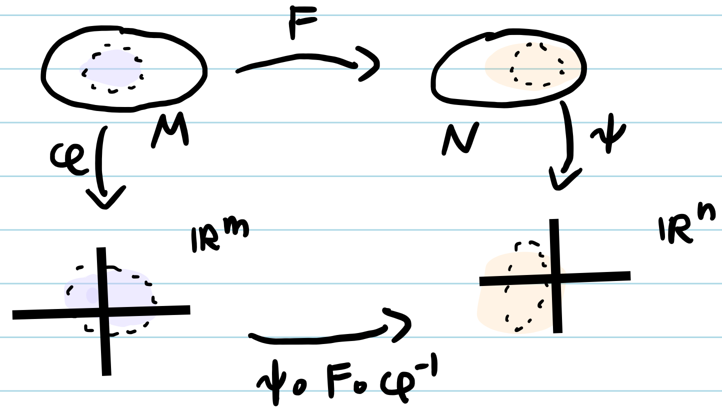
$$h(\tilde{u}, \tilde{v}) \overset{\text{loc}}{=} \sqrt{1-(\tilde{u})^2-(\tilde{v})^2}$$

must use coordinates belonging to given chart.
Same function can look different in different charts.

Def: M, N smooth mfd

$F: M \to N$ is smooth if $\psi \circ F \circ \varphi^{-1}$ is smooth

$\forall\, (\mathcal{U}, \varphi) \in \mathcal{A}$ atlas for M with $F(\mathcal{U}) \cap V \neq \phi$.
$(V, \psi) \in \mathcal{A}'$ atlas for N



Def: M and N are diffeomorphic if
$\exists F: M \to N$ bijection s.t. F and F^{-1} are smooth.

ex) $F: S^1 \to S^1$ antipodal map

$F(e^{i\theta}) = -e^{i\theta}$

Inverse $=$ F itself

Over $\mathcal{U} = \bigcirc = \{ e^{i\theta} : 0 < \theta < 2\pi \}$

$F(\theta) \overset{loc}{=} \theta + \pi$

Over $\tilde{\mathcal{U}} = \bigcirc = \{ e^{i\tilde{\theta}} : -\pi < \tilde{\theta} < \pi \}$

$F(\tilde{\theta}) \overset{loc}{=} \tilde{\theta} + \pi$

ex) $\mathbb{CP}^n = \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \sim$

points denoted: $[z_0, \cdots, z_n] \in \mathbb{CP}^n$

$[z_0, \cdots, z_n] \sim [x_0, \cdots, x_n] \iff \begin{pmatrix} z_0 \\ \vdots \\ z_n \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}, \quad \lambda \in \mathbb{C}^*$

$\mathbb{CP}^n$ is a mfd of dim $2n$
For now: we focus on $\mathbb{CP}^1$.

Charts: $\mathbb{CP}^1 = \mathcal{U}_0 \cup \mathcal{U}_1$

$\mathcal{U}_0 = \{ z_0 \neq 0 \}$
coords: $z = x + iy$ with $z = \frac{z_1}{z_0}$.

$\mathcal{U}_1 = \{ z_1 \neq 0 \}$
coords: $\tilde{z} = \tilde{x} + i\tilde{y}$ with $\tilde{z} = \frac{z_0}{z_1}$.

Change of coords: on $\mathcal{U}_0 \cap \mathcal{U}_1$, $\tilde{z} = 1/z$

$$\tilde{x} + i\tilde{y} = \frac{x - iy}{x^2 + y^2}$$

$\begin{cases} \tilde{x} = \dfrac{x}{x^2+y^2} \\ \tilde{y} = \dfrac{y}{x^2+y^2} \end{cases} \implies \mathbb{CP}^1$ is a smooth mfd of dim $= 2$.

ex) $F: S^2 \to \mathbb{CP}^1$

$$F(x,y,z) = \begin{cases} [x+iy, 1-z] & \text{if } z \neq 1 \\ [1, 0] & \text{if } (x,y,z) = (0,0,1) \end{cases}$$

is a diffeomorphism.

injective: if $(x_1 + iy_1, 1 - z_1) = \lambda(x_2 + iy_2, 1 - z_2)$, $\lambda \in \mathbb{C}^*$, $(x_i, y_i, z_i) \in S^2$
then $(x_1, y_1, z_1) = (x_2, y_2, z_2)$.

surjective: given $(a, b) \in \mathbb{C}^2$, $b \neq 0$, can solve:

$$(x + iy, 1 - z) = \lambda(a, b), \quad \lambda \in \mathbb{C}^*, \ (x, y, z) \in S^2.$$

F smooth: need to check in local charts.

e.g. $\{x > 0\} \subseteq S^2$, $\mathcal{U}_1 \subseteq \mathbb{C}\mathbb{P}^1$

$$F(y, z) \overset{loc}{=} \frac{(\sqrt{1 - y^2 - z^2}, y)}{1 - z}$$

smooth over domain $= \{y^2 + z^2 < 1\} = \{x > 0\} \cap S^2$

e.g. $\{z > 0\} \cap \{z \neq 1\} \subseteq S^2$, $\mathcal{U}_0 \subseteq \mathbb{C}\mathbb{P}^1$

$$F(x, y, z) = \frac{x - iy}{1 - z}[x + iy, 1 - z] = \left[\frac{x^2 + y^2}{1 - z}, x - iy\right]$$
$$= [1 + z, x - iy]$$

$$F(x, y) \overset{loc}{=} \frac{(x, -y)}{1 + (1 - x^2 - y^2)^{\frac{1}{2}}}$$

smooth over $\{x^2 + y^2 < 1\}$ $\Rightarrow$ F smoothly extends through $(x, y, z) = (0, 0, 1)$.

Similarly: can check all chart combos give smooth local functions.

F^{-1} smooth: ... omitted. Need to work out F^{-1} and look in local coords.