3. Tangent Vectors

Def: Let M be smooth mfd with $\dim M = n$.
: Let $p \in M$.
A tangent vector $V \in T_pM$ is defined by the following:

For each chart $(\mathcal{U}, \varphi, x^i) \in \mathcal{A}$ with $p \in \mathcal{U}$, assign a vector

$$\begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix} \in \mathbb{R}^n \qquad \text{such that}$$

if: over $(\mathcal{U}, \varphi, x^i)$, assign $(v^1, \dots, v^n)$
over $(\tilde{\mathcal{U}}, \tilde{\varphi}, \tilde{x}^i)$, assign $(\tilde{v}^1, \dots, \tilde{v}^n)$

then: $$\tilde{v}^i = \sum_{k=1}^{n} \frac{\partial \tilde{x}^i}{\partial x^k} v^k \quad \forall i.$$

Common notation:

1) Summation convention: $\tilde{v}^i = \dfrac{\partial \tilde{x}^i}{\partial x^k} v^k$ omit Σ for repeated indices

2) $V \in T_pM$, write $V \overset{\text{loc}}{=} v^i \dfrac{\partial}{\partial x^i}\Big|_p$ over $(\mathcal{U}, x^i)$.
summation

Reason for notation 2)

$V \in T_pM$ defines a derivation $V : C^\infty(M) \to \mathbb{R}$
by
$$V(f) := v^i \frac{\partial}{\partial x^i} f.$$

Why is $V(f)$ well-defined?
If: $V \overset{\text{loc}}{=} v^i \dfrac{\partial}{\partial x^i}$ over $(\mathcal{U}, x)$, $V \overset{\text{loc}}{=} \tilde{v}^i \dfrac{\partial}{\partial \tilde{x}^i}$ over $(\tilde{\mathcal{U}}, \tilde{x})$
then: ...

$$V^i \frac{\partial}{\partial x^i} f \overset{\text{chain rule}}{=} V^i \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial}{\partial \tilde{x}^p} f \overset{\text{defn } \tilde{V}^p}{=} \tilde{V}^p \frac{\partial}{\partial \tilde{x}^p} f .$$

ex) $M = \mathbb{R}^2 \setminus \{0\}$ with charts:

$(\mathcal{U}, (x,y))$ Cartesian coords, $\mathcal{U} = M$

$(\tilde{\mathcal{U}}, (r,\theta))$ Polar coords, $\tilde{\mathcal{U}} = M$.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Define $V \overset{\text{loc}}{=} \left.\frac{\partial}{\partial \theta}\right|_p$ over $\tilde{\mathcal{U}}$. $p = (x,y) \in M$ fixed.

How does V appear over $(\mathcal{U}, x)$?

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial}{\partial y}$$

$$= -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}$$

$$\Rightarrow V \overset{\text{loc}}{=} -y \left.\frac{\partial}{\partial x}\right|_p + x \left.\frac{\partial}{\partial y}\right|_p \text{ over } \mathcal{U}.$$

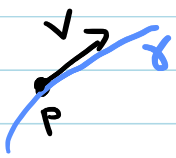
Other notation:

$V \overset{\text{loc}}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ over $\tilde{\mathcal{U}}$, $\quad V \overset{\text{loc}}{=} \begin{pmatrix} -y \\ x \end{pmatrix}$ over $\mathcal{U}$.

Velocity Vector of Curves

Def: A curve in M is a smooth map $\gamma : J \to M$, $J \subseteq \mathbb{R}$ interval

Prop: Every $V \in T_pM$ can be written as



$$V = \frac{d\gamma}{dt}(0) \text{ for some curve } \gamma : (-\varepsilon, \varepsilon) \to M \text{ with } \gamma(0) = p.$$

Pf: ① Why does $\frac{d\gamma}{dt}(0)$ define a tangent vector?

Let $\gamma: (-\varepsilon, \varepsilon) \to M$ be s.t. $\gamma(0) = p$.
Let $(\mathcal{U}, x)$, $(\tilde{\mathcal{U}}, \tilde{x})$ be coords near p

Over $\mathcal{U}$: $\gamma(t) \overset{loc}{=} \begin{pmatrix} x^1(t) \\ \vdots \\ x^n(t) \end{pmatrix}$, $V^i \overset{loc}{=} \frac{dx^i}{dt}(0)$.

over $\tilde{\mathcal{U}}$: $\gamma(t) \overset{loc}{=} \begin{pmatrix} \tilde{x}^1(t) \\ \vdots \\ \tilde{x}^n(t) \end{pmatrix}$, $\tilde{V}^i \overset{loc}{=} \frac{d\tilde{x}^i}{dt}(0)$

Chain rule $\Rightarrow$ $\frac{d\tilde{x}^i}{dt} = \frac{\partial \tilde{x}^i}{\partial x^p} \frac{dx^p}{dt}$.

$\Rightarrow$ $\tilde{V}^i = \frac{\partial \tilde{x}^i}{\partial x^p} V^i$ ✓ defines $V \in T_pM$.

② Given $V \in T_pM$, why does it come from $V = \frac{d\gamma}{dt}(0)$?

Let $p \in (\mathcal{U}, x)$, write $V \overset{loc}{=} V^i \frac{\partial}{\partial x^i}$ and define

$$\gamma(t) = \begin{pmatrix} p^1 \\ \vdots \\ p^n \end{pmatrix} + t \begin{pmatrix} V^1 \\ \vdots \\ V^n \end{pmatrix}.$$

$\square$

ex) Calc III. surface $\{G(x,y,z) = 0\} \subseteq \mathbb{R}^3$
e.g. $x^2+y^2+z^2-1 = 0$.

T_pM = tangent plane at p = plane through p with normal $\nabla G|_p$.

Let $V \in T_pM$, write $V = \frac{d\gamma}{dt}(0)$, $\gamma(0) = p$.

$$\frac{d}{dt} G(\gamma(t)) = 0$$

$$\nabla G|_p \cdot \frac{d\gamma}{dt}(0) = 0 \quad \Rightarrow \text{All } V \in T_pM \text{ satisfy } \nabla G|_p \cdot V = 0.$$

$$\underset{\text{dim } 2}{T_pM} \subseteq \underset{\text{dim} 2}{\{ v \in \mathbb{R}^3 : \nabla G|_p \cdot v = 0 \}}$$

$$\Rightarrow T_pM = \{ v \in \mathbb{R}^3 : \nabla G|_p \cdot v = 0 \}$$

Def: (Push forward) M, $\dim M = m$
N, $\dim N = n$

$F : M \to N$ smooth map, $p \in M$.

Define: $dF_p : T_pM \to T_{F(p)}N$ by:

① choose coords: $p \in (\mathcal{U}, x^i) \subseteq M$
$F(p) \in (V, y^i) \subseteq N$

and write $F \overset{loc}{=} \begin{pmatrix} F^1(x) \\ \vdots \\ F^n(x) \end{pmatrix}$

② define: $dF_p(V) = \frac{\partial F^k}{\partial x^i}(p) V^i \frac{\partial}{\partial y^k}\Big|_p$.

More notation:

a) $dF_p\left(\frac{\partial}{\partial x^i}\Big|_p \right) = \frac{\partial F^k}{\partial x^i} \frac{\partial}{\partial y^k}\Big|_p$

b) $dF_p \overset{loc}{=} \begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \frac{\partial F^1}{\partial x^2} & \cdots \\ \frac{\partial F^2}{\partial x^1} & \frac{\partial F^2}{\partial x^2} & \ddots \\ \vdots & \vdots & \end{pmatrix} = \left(\frac{\partial F^k}{\partial x^i} \right)_{n \times m}$

c) If $v \in T_pM$, then $dF_p(v) \in T_{F(p)}N$ so acts on $C^\infty(N)$:

$dF_p(v)(f) = v(f \circ F).$

$= v^i \frac{\partial}{\partial x^i} (f(F(x)))$

$= v^i \frac{\partial F^k}{\partial x^i} \frac{\partial}{\partial y^k} f$

chain rule.

To verify $dF_p(v) \in T_{F(p)}N$, need to check:

$\frac{\partial \tilde{F}^k}{\partial x^i} = \frac{\partial \tilde{y}^k}{\partial x^i} \frac{\partial F^p}{\partial x^i}$ for charts: $(\mathcal{U}, \varphi, x)$ on M
(V, ψ, y) on N
$(\tilde{V}, \tilde{\psi}, \tilde{y})$ on N

where: $\tilde{\psi} \circ \psi^{-1}(y) = (\tilde{y}^1(y), \dots, \tilde{y}^n(y))$

$\tilde{F}^k(x) = \tilde{\psi}^k \circ F \circ \varphi^{-1}(x)$

$F^k(x) = \psi^k \circ F \circ \varphi^{-1}(x)$

Indeed: $\frac{\partial \tilde{F}^k}{\partial x^i} = \frac{\partial}{\partial x^i} \tilde{\psi}^k \circ \psi^{-1} \circ F \circ \varphi^{-1}(x)$

$= \frac{\partial}{\partial x^i} \tilde{y}^k \circ F(\varphi^{-1}(x))$

$= \frac{\partial \tilde{y}^k}{\partial y^p} \frac{\partial F^p}{\partial x^i}$ chain rule

Computing dF using a path

Prop: $F: M \to N$ smooth map
$p \in M$
$V \in T_pM$

$$dF_p(V) = \left.\frac{d}{dt}\right|_{t=0} F\circ\gamma(t) \quad \text{for any path } \gamma: (-\varepsilon,\varepsilon)\to M$$

with: $\gamma(0) = p$
$\frac{d}{dt}\gamma(0) = V.$

Pf: Chain rule in coords:

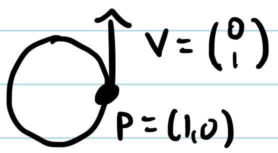
$$\left.\frac{d}{dt}\right|_{t=0} F\circ\gamma(t) = \frac{\partial F^k}{\partial x^i}\underbrace{\frac{d\gamma^i(0)}{dt}}_{V^i}\frac{\partial}{\partial y^k} = dF|_p(V).$$

$\square$

ex) $S^1 = \{x^2+y^2=1\} \subseteq \mathbb{R}^2$

$F(x,y) = (-y,x).$
$F: S^1 \to S^1.$

$\begin{pmatrix}0\\1\end{pmatrix} \in T_{(1,0)}S^1$: comes from path $\gamma(t) = (\cos t, \sin t)$
$\gamma(0) = (1,0)$
$\dot\gamma(0) = \begin{pmatrix}0\\1\end{pmatrix}$

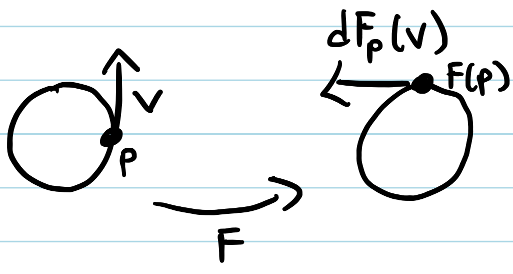


Compute $dF_{(1,0)}\begin{pmatrix}0\\1\end{pmatrix} = ?$

$$dF_{(1,0)}\begin{pmatrix}0\\1\end{pmatrix} = \left.\frac{d}{dt}\right|_{t=0} F\circ\gamma(t)$$

$$= \left.\frac{d}{dt}\right|_{t=0} (-\sin t, \cos t)$$

$$= \begin{pmatrix}-1\\0\end{pmatrix}.$$



Tangent Bundle

$$\begin{cases} M \text{ mfd, } \dim M = n \end{cases} \rightsquigarrow \begin{cases} TM \text{ mfd, } \dim TM = 2n \\ TM = \bigsqcup_{p \in M} T_p M \end{cases}$$

$\pi : TM \to M, \quad \pi(p, v) = p$

$p \in M \qquad v \in T_pM$

Charts on TM :

- Chart $(\mathcal{U}, \varphi, x)$ for M $\rightsquigarrow$ chart $(\pi^{-1}(\mathcal{U}), \tilde{\varphi})$ for TM.

$\tilde{\varphi} : \pi^{-1}(\mathcal{U}) \to \mathbb{R}^{2n}$

$$\tilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\Big|_p \right) = (x^1(p), \dots, x^n(p), v^1, \dots, v^n).$$

change of coords:

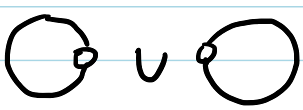
- coords $(\mathcal{U}, x^i), (\tilde{\mathcal{U}}, \tilde{x}^i)$ on M

  $\tilde{x}^i = f^i(x^1, \dots, x^n)$ change of coords on M

- coords $(\pi^{-1}(\mathcal{U}), (x^i, v^i)), (\pi^{-1}(\tilde{\mathcal{U}}), (\tilde{x}^i, \tilde{v}^i))$ on TM

$$\begin{cases} \tilde{x}^i = f^i(x^1, \dots, x^n) \\ \tilde{v}^i = \dfrac{\partial \tilde{x}^i}{\partial x^p} v^p \end{cases} \quad \text{change of coords on } TM$$

ex) $S^1 = \{ e^{i\theta} : \theta \in [0, 2\pi] \}$

$S^1 = \mathcal{U} \cup \tilde{\mathcal{U}} =$ 

$\mathcal{U} = \{ e^{i\theta} : \theta \in (0, 2\pi) \}$ coord θ

$\tilde{\mathcal{U}} = \{ e^{i\tilde{\theta}} : \tilde{\theta} \in (-\pi, \pi) \}$ coord $\tilde{\theta}$

Compute change of coords on TS^1 and show:

$$TS^1 = S^1 \times \mathbb{R}.$$

ex) $S^n \subseteq \mathbb{R}^{n+1}$

$G : \mathbb{R}^{n+1} \to \mathbb{R}$,

$G(x) = |x|^2$.

$S^n = \{ G = 1 \}$, $T_p S^n = \{ v \in \mathbb{R}^{n+1} : \nabla G(p) \cdot v = 0 \}$

$\Rightarrow TS^n = \{ (x, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : |x| = 1 \text{ and } x \cdot v = 0 \}$