4. Submersions, Immersions, Embeddings

Inverse Function Theorem

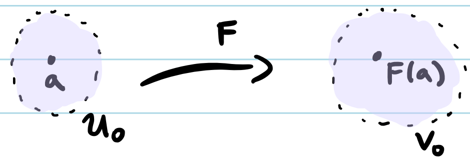
$\mathcal{U}, V \subseteq \mathbb{R}^n$ open

$F : \mathcal{U} \to V$ smooth with $DF|_a$ invertible, $a \in \mathcal{U}$.

$\Rightarrow \exists\ \mathcal{U}_0 \subseteq \mathcal{U}$, $a \in \mathcal{U}_0$

$V_0 \subseteq V$, $F(a) \in V_0$ s.t. $F : \mathcal{U}_0 \to V_0$ is a diffeomorphism.

(Recall $DF = \left(\dfrac{\partial F^i}{\partial x^j} \right)_{n \times n}$).



We will assume the inverse function thm, and derive:

(A) Implicit Function Thm

(B) Rank Thm

Implicit Function Theorem

$\mathcal{U}_1 \subseteq \mathbb{R}^n$, $\mathcal{U}_2 \subseteq \mathbb{R}^k$ open

$F : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathbb{R}^k$ smooth

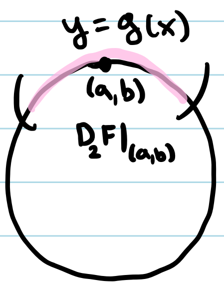
$(x, y) \mapsto F(x,y)$

Let $(a,b) \in \mathcal{U}_1 \times \mathcal{U}_2$ be s.t. $F(a,b) = 0$.

Assume $\left(\dfrac{\partial F^i}{\partial y^j} \right) \Big|_{(a,b)}$ is invertible.

$$\Rightarrow \exists \begin{cases} V_1 \subseteq \mathcal{U}_1, & a \in V_1 \\ V_2 \subseteq \mathcal{U}_2, & b \in V_2 \\ g : V_1 \to V_2 & \text{smooth} \end{cases} \quad \text{s.t. } \forall (x,y) \in V_1 \times V_2,\ F(x,y) = 0 \iff y = g(x).$$

In other words, can find parameters x for the set $\{F = 0\}$ s.t. $F(x, g(x)) = 0$.



$F(x,y) = 0$

e.g. $F(x,y) = x^2 + y^2 - 1$

$\partial_y F = 2y \neq 0$ if $y \neq 0$

Pf: Consider $\varphi(x,y) = (x, F(x,y))$.

$$D\varphi\Big|_{(a,b)} = \begin{pmatrix} I_{n\times n} & 0 \\ \frac{\partial F^i}{\partial x^j}\Big|_{(a,b)} & \frac{\partial F^i}{\partial y^j}\Big|_{(a,b)} \end{pmatrix} \quad \text{invertible.}$$

IFT $\Rightarrow$ $\exists$ local inverse $\varphi^{-1}(x,y)$.

Check: $\varphi^{-1}(x,y) = (x, A(x,y))$ $\quad \varphi(\varphi^{-1}(x,y)) = (x,y)$

Define: $g(x) = A(x,0)$.

(*) explained below

Then $F(x, g(x)) = F(x, A(x,0)) = 0$.

(*) $\varphi(\varphi^{-1}(x,y)) = (x,y)$
$\Leftrightarrow \varphi(x, A(x,y)) = (x,y)$
$\Leftrightarrow (x, \underline{F}(x, A(x,y))) = (x, \underline{y})$

Conversely, if $F(x,y) = 0$, (x,y) near (a,b),
$\Rightarrow \varphi(x,y) = (x,0)$
$\Rightarrow \varphi^{-1}(\varphi(x,y)) = \varphi^{-1}(x,0)$
$\quad (x, \underline{y}) = (x, \underline{g(x)}) \quad \Rightarrow y = g(x)$. $\square$

Def: M, N sm mfd, $p \in M$
$F: M \to N$ smooth
rank of F at p = rank of $dF\big|_p : T_pM \to T_{F(p)}N$.

Rank Theorem

$W \subseteq \mathbb{R}^m$ open
$F: W \to \mathbb{R}^n$ smooth with constant rank k
$\forall p \in W$, $\exists$ charts $(\mathcal{U}, \varphi)$ centered at p such that
(V, ψ) centered at $F(p)$

$$\psi \circ F \circ \varphi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^k, 0, \ldots, 0).$$

ex) $F(x,y,z) = x^2+y^2+z^2$. Near $(a,b,c) \neq (0,0,0)$, F has a coord representation of the form $F(u,v,w) = u$.

ex) $F: M^m \to N^n$, $m \geq n$, $\operatorname{rank}(F) \equiv n$, then locally $F(x^1,\dots,x^n,y^1,\dots,y^{m-n}) = (x^1,\dots,x^n)$.

ex) $F: M^m \to N^n$, $m \leq n$, $\operatorname{rank}(F) \equiv m$, then locally $F(x^1,\dots,x^m) = (x^1,\dots,x^m,0,\dots,0)$.

Pf: For convenience, assume $p=0$, $F(p)=0$,

$$F(\underset{\in\mathbb{R}^k}{x}, \underset{\in\mathbb{R}^{m-k}}{y}) = (\underset{\in\mathbb{R}^k}{Q(x,y)}, \underset{\in\mathbb{R}^{n-k}}{R(x,y)}), \quad \left.\frac{\partial Q^i}{\partial x^j}\right|_0 \text{ invertible.}$$

$\overbrace{x,y}^{\in\mathbb{R}^m}$

Consider $\varphi(x,y) = (Q(x,y), y)$.

$$D\varphi|_0 = \begin{pmatrix} \left.\frac{\partial Q^i}{\partial x^j}\right|_0 & \left.\frac{\partial Q^i}{\partial y^j}\right|_0 \\ 0 & I \end{pmatrix} \quad \text{invertible.}$$

IFT $\Rightarrow \exists$ local inverse of the form $\varphi^{-1}(x,y) = (A(x,y), y)$ with $Q(A(x,y),y) = x$.

$\varphi \circ \varphi^{-1} = (x,y)$

$$\Rightarrow F\circ\varphi^{-1}(x,y) = (x, \underbrace{R(A(x,y),y)}_{\tilde{R}(x,y)}).$$

$$D(F\circ\varphi^{-1}) = \begin{pmatrix} I_{k\times k} & 0 \\ \frac{\partial\tilde{R}}{\partial x} & \frac{\partial\tilde{R}}{\partial y} \end{pmatrix}$$

must have rank k since: φ is diffeo, F rank k.

$\Rightarrow \frac{\partial\tilde{R}^i}{\partial y^j} \equiv 0$. $\Rightarrow \tilde{R}$ does not depend on y.

$\Rightarrow F\circ\varphi^{-1}(x,y) = (x, S(x))$, $\quad S(x) = \tilde{R}(x,0)$.

Introduce $\psi: (u,v) \mapsto (u, v-S(u))$.

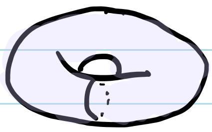
$\Rightarrow \psi\circ F\circ\varphi^{-1}: (x,y) \mapsto (x,0)$. $\square$

Immersion: Smooth map $F: M \to N$ with $dF|_p$ injective $\forall p \in M$.

ex) $\gamma: (a,b) \to M$ smooth curve with $\dot{\gamma}(t) \neq 0 \ \forall t$.

ex) $f: \mathbb{R}^2 \to \mathbb{R}^3$

$$f(\varphi, \theta) = \big((1+\cos\varphi)\cos\theta, (1+\cos\varphi)\sin\theta, \sin\varphi \big) \quad (*)$$

 $\subseteq \mathbb{R}^3$ Exercise: compute df.

Submersion: Smooth map $F: M \to N$ with $dF|_p$ surjective $\forall p \in M$.

ex) $F: \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$
$F(x,y,z) = x^2 + y^2 + z^2$.

ex) $\pi: TM \to M$.
(recall: locally $\pi(x^i, v^i) = x^i$)

Embedding: $F: M \to N$ immersion which is homeomorphism onto its image $F(M) \subseteq N$.

ex) $f: S^1 \times S^1 \to \mathbb{R}^3$ in $(*)$ is an embedding of the torus into $\mathbb{R}^3$.

Exercise: 1) Show f is immersion
2) Show f is injective
3) Use "closed map thm" from topology:

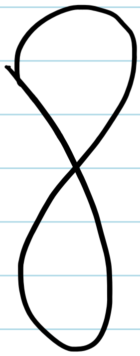
Closed map thm: Suppose: X compact
Y Hausdorff
$F: X \to Y$ cont and bijective

Then: $F: X \to Y$ is a homeomorphism
(i.e. F^{-1} is continuous)

ex) $\gamma : (-\pi, \pi) \to \mathbb{R}^3$

$\gamma(t) = (\sin 2t, \sin t)$ is a immersion, but not an embedding since : $(-\pi,\pi)$ not compact

$\gamma(-\pi,\pi)$ is compact



ex) $\gamma : \mathbb{R} \to T^2$

$\gamma(t) = (e^{2\pi i t}, e^{2\pi i \alpha t})$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$

γ is an immersion, but not an embedding since $\gamma(\mathbb{R})$ is dense in T^2.