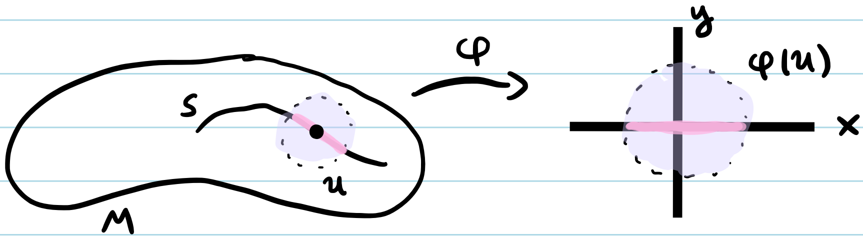
5. Submanifolds

- Let M be a smooth mfd, $\dim M = m$.

Def: $S \subseteq M$ is a submanifold of dimension k if: $\forall x \in S$, $\exists$ chart $(\mathcal{U}, \varphi)$ on M with $x \in \mathcal{U}$ s.t.

$$\varphi(S \cap \mathcal{U}) = \{ (x^1, \dots, x^k, y^{k+1}, \dots, y^m) \in \varphi(\mathcal{U}) : y^{k+1} = \dots = y^m = 0 \}.$$



Notation: Drop the φ, just write

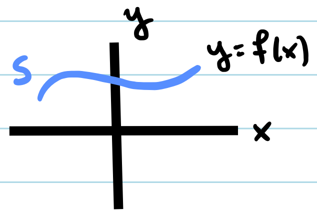
$$S \cap \mathcal{U} = \{ (x^1, \dots, x^k, y^1, \dots, y^{m-k}) \in \mathcal{U} : y^i = 0 \ \forall i \}$$

Remark: A submanifold is a manifold.

Proof: Lee Thm 5.8. $\square$

ex) $S = \{ (x, f(x)) : x \in \mathbb{R}^n \} \subseteq \mathbb{R}^{n+1}$, $\quad f : \mathbb{R}^n \to \mathbb{R}$ smooth function

Let $\tilde{x}^1 = x^1$ $\quad \Rightarrow S = \{ \tilde{y} = 0 \}$.
$\vdots$
$\tilde{x}^n = x^n$
$\tilde{y} = x^{n+1} - f(x)$ $\quad \therefore S$ submfd of $\mathbb{R}^{n+1}$.



Constant-Rank Level Set Thm: M, N smooth manifolds
$F : M \to N$ smooth map with constant rank
$\Rightarrow F^{-1}(c) \subseteq M$ is a submanifold $\forall c \in F(M)$.

Pf: Let $p \in F^{-1}(c)$. By the rank thm: $\exists$ coords with $p = 0$, $F(p) = 0$, $c = 0$, $\mathcal{U} \subseteq M$ open s.t.

$$F(y^1, \dots, y^r, x^1, \dots, x^{m-r}) = (y^1, \dots, y^r, 0, \dots, 0)$$

$$F^{-1}(0) \cap \mathcal{U} = \{ (y^1, \dots, y^r, x^1, \dots, x^{m-r}) \in \mathcal{U} : y^1 = \dots = y^r = 0 \}.$$

$\square$

Def: Let $F: M \to N$ be a smooth map.

1) $p \in M$ is a regular pt if
$dF_p : T_pM \to T_{F(p)}N$ is surjective.

2) $p \in M$ is a critical pt if
$dF_p : T_pM \to T_{F(p)}N$ is <u>not</u> surjective.

3) $c \in N$ is a regular value if every $p \in F^{-1}(c)$ is a regular point.

Note: If $F^{-1}(c) = \emptyset$, then c is a regular value.

Regular Level Set Thm: $F: M \to N$ smooth map. $\dim M = m$, $\dim N = n$.
$c \in N$ regular value, $c \in F(M)$.
$\Rightarrow F^{-1}(c) \subseteq M$ is a submanifold of dim: $m-n$.

Pf: Let $p \in F^{-1}(c)$. Then $\operatorname{rank}(dF_p) = n = \dim N$.

Claim: $\exists\, \mathcal{U} \subseteq M$ open with $p \in \mathcal{U}$ s.t. $\operatorname{rank}(dF_q) \equiv n \quad \forall q \in \mathcal{U}$.

Assuming claim: $F: \mathcal{U} \to N$ has constant rank and so
$$F^{-1}(c) \cap \mathcal{U} = \{ (x^1, \dots, x^{m-n}, y^1, \dots, y^n) \in \mathcal{U} : y^i = 0 \ \ \forall i \}$$
$\therefore F^{-1}(c)$ is submfd.

Explanation of claim: Set of $m \times n$ matrices of full rank is an open set.
Full rank = Some submatrix has $\det \neq 0$
$\det \neq 0$: open condition.

$\square$

ex) $F: \mathbb{R}^n \to \mathbb{R}$
$$F(x^1, \dots, x^n) = \sum_{i=1}^{n} (x^i)^2.$$

$dF_x = (2x^1, 2x^2, \dots, 2x^n)$ surjective except at $x = 0$.

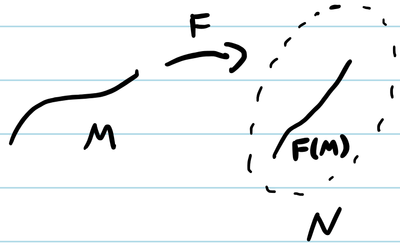
$F^{-1}(\delta) =$ Sphere radius $\delta > 0$
$F^{-1}(\delta) \subseteq \mathbb{R}^n$ submanifold

Prop: If $F: M^m \to N^n$ is embedding, then: $F(M) \subseteq N$ submanifold.

Pf: Embedding means: dF_p injective $\forall p$ and
$\quad$ rank m
$F: M \to F(M)$ homeomorphism ($F(M) \subseteq N$ subspace topology)

Rank Thm: $\forall p \in M$, $\exists$ nbhd $\mathcal{U}$ with coords s.t.

$$F(x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0).$$



Homeo $\Rightarrow$ $F(\mathcal{U})$ open in subspace topology.
$\Rightarrow$ $F(\mathcal{U}) = F(M) \cap V$, $\quad V \subseteq N$ open.

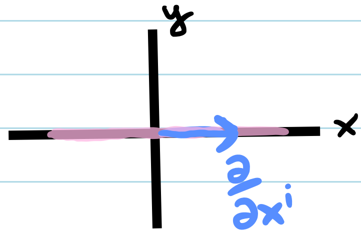
$$\therefore\ F(M) \cap V = \{ (x^1, \dots, x^m, y^1, \dots, y^{n-m}) \in V : y^i = 0 \ \forall i \}. \qquad \square$$

Tangent Space of a Submanifold

Let $S \subseteq M$ be a submfd. In local coords:

$$S \cap \mathcal{U} = \{ (x^1, \dots, x^k, y^1, \dots, y^{n-k}) \in \mathcal{U} \text{ s.t. } y^i = 0 \ \forall i \}$$

$$T_p S = \text{span} \left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^k}\right|_p \right\}$$



Prop: Suppose $F: M^m \to N^n$, $c = F(p)$ regular value.
$S = F^{-1}(c)$ submfd of dim $m-n$.
$\Rightarrow T_p S = \operatorname{Ker} dF_p \subseteq T_p M.$

Pf: $T_p S = \{ \dot{\gamma}(0) : \gamma : (-\varepsilon, \varepsilon) \to S \text{ with } \gamma(0) = p \} \subseteq T_p M$

$c = F(\gamma(t))$

$$0 = \left.\frac{d}{dt}\right|_{t=0} F(\gamma(t)) \Rightarrow 0 = \frac{\partial F^i}{\partial x^k} \frac{d\gamma^k}{dt}(0) \ \forall i \Rightarrow 0 = dF_p(\dot{\gamma}(0)).$$

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \frac{\partial F^1}{\partial x^2} \\ \frac{\partial F^2}{\partial x^1} & \frac{\partial F^2}{\partial x^2} \end{pmatrix} \begin{pmatrix} \dot{\gamma}^1(0) \\ \dot{\gamma}^2(0) \end{pmatrix}$$

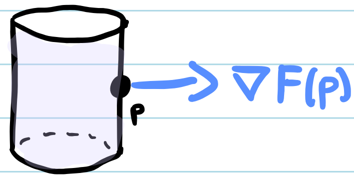
$m = \dim \ker dF_p + \underbrace{\dim \operatorname{im} dF_p}_{= n}$
surjective

$\therefore \underbrace{\operatorname{Ker} dF_p}_{\dim = m-n} \supseteq \underbrace{T_p S}_{\dim = m-n} \Rightarrow \operatorname{Ker} dF_p = T_p S. \qquad \square$

ex) $F : \mathbb{R}^3 \to \mathbb{R}$

$S : \{ F(x,y,z) = c \} \subseteq \mathbb{R}^3$

$T_pS = \{ v \in \mathbb{R}^3 : \underbrace{\nabla F(p) \cdot v}_{dF_p(v)} = 0 \}$



$F(x,y,z) = 0$

ex) $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$

$\qquad A \mapsto \det A$

$SL(n,\mathbb{R}) = \det^{-1}(1).$

$\qquad = \{ A \in M_{n\times n}(\mathbb{R}) : \det A = 1 \}$

$M_{n\times n}(\mathbb{R})$ is mfd of dim n^2. (diffeo to $\mathbb{R}^{n^2}$)

What is $T_A M_{n\times n}(\mathbb{R})$, $A \in M_{n\times n}(\mathbb{R})$.

e.g. 2x2 matrices, coords $B = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$.

$T_A M_{n\times n}(\mathbb{R}) = \text{span}\left\{ \frac{\partial}{\partial x_{11}}\Big|_A, \frac{\partial}{\partial x_{12}}\Big|_A, \frac{\partial}{\partial x_{21}}\Big|_A, \frac{\partial}{\partial x_{22}}\Big|_A \right\}$

but instead we identify

$\frac{\partial}{\partial x_{11}} \leftrightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\frac{\partial}{\partial x_{12}} \leftrightsquigarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\frac{\partial}{\partial x_{21}} \leftrightsquigarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\frac{\partial}{\partial x_{22}} \leftrightsquigarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

So $T_A M_{n\times n}(\mathbb{R}) = \text{Span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$\qquad \cong M_{n\times n}(\mathbb{R}).$ Tangent vector on $M_{n\times n}$ can be represented by a matrix.

To show $SL(n,\mathbb{R})$ is a manifold of dim n^2-1: show 1 is a regular value of $\det : M_{n\times n} \to \mathbb{R}$.

Compute $d(\det)_A$ using a path:

Let $A \in SL(n,\mathbb{R})$, $\gamma(t) = A + tM$ where $M \in M_{n\times n}(\mathbb{R})$.

$\gamma(0) = A \in SL(n,\mathbb{R}) \subseteq M_{n\times n}(\mathbb{R})$

$\dot\gamma(0) = M \in T_A M_{n\times n}(\mathbb{R})$

$\gamma(0) = p$, $\underset{=v}{\dot\gamma(0)}$ $\quad dF_p(v) = \frac{d}{dt}\Big|_{t=0} F(\gamma(t))$

$$d(\det)_A(M) = \frac{d}{dt}\Big|_{t=0} \det(A+tM)$$

$$= \underbrace{(\det A)}_{=1} \operatorname{Tr} A^{-1}M$$ will prove below

$d(\det)_A$ is surj: given $c \in \mathbb{R}$, then

$d(\det)_A(cA) = \operatorname{Tr} A^{-1}(cA) = c$ ✓

<u>Jacobi's formula</u>: $\frac{d}{dt}\Big|_{t=0} \det(A+tM) = (\det A) \operatorname{Tr} A^{-1}M$

<u>Pf</u>: 1) $\det(I+tB) = 1 + (\operatorname{Tr} B)t + O(t^2)$ e.g. check for 2×2 matrices

$\Rightarrow \frac{d}{dt}\Big|_{t=0} \det(I+tB) = \operatorname{Tr} B.$

2) $\frac{d}{dt}\Big|_{t=0} \det(A+tM) = (\det A)\frac{d}{dt}\Big|_{t=0} \det(I+tA^{-1}M) = (\det A) \operatorname{Tr} A^{-1}M.$

$\therefore SL(n,\mathbb{R}) \subseteq M_{n\times n}(\mathbb{R})$ is a submanifold of dim n^2-1.

$$T_A SL(n,\mathbb{R}) = \operatorname{Ker} d(\det)_A$$

$$= \operatorname{Ker}\{M \mapsto \operatorname{Tr} A^{-1}M\}$$

$\therefore T_A SL(n,\mathbb{R}) = \{M \in M_{n\times n}(\mathbb{R}) : \operatorname{Tr} A^{-1}M = 0\}.$