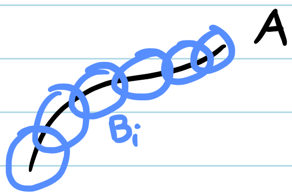
6. Sard's Theorem

Def: $A \subseteq \mathbb{R}^n$ has measure zero if $\forall \delta > 0$, $\exists \{B_i\}$ countable collection of balls $B_i = B_{r_i}(p_i)$ s.t.



1) $A \subseteq \bigcup B_i$
2) $\sum_i \mathrm{Vol}(B_i) < \delta$.

Lemma: Let $I^m = [0,1]^m \subseteq \mathbb{R}^m$ be unit cube
$F: I^m \to \mathbb{R}^m$ C^1 map
$A \subseteq I^m$ measure zero

Then: $F(A)$ has measure zero.

Def: $A \subseteq M$ subset of a mfd has measure zero if $\forall$ charts $(\mathcal{U}, \varphi)$ then $\varphi(A \cap \mathcal{U}) \subseteq \mathbb{R}^m$ has measure zero.

Fact: Measure zero property is indep of choice of atlas.

Proof of Lemma: $\exists M > 0$ s.t.

$$|F(x) - F(y)| \leq M|x-y| \quad \forall x,y \in I^m. \quad (f \in C^1(I^m))$$

$\Rightarrow$ If $y \in B_r(x)$ $(|x-y| < r)$ then $|F(x) - F(y)| \leq Mr \Rightarrow F(B_r(x)) \subseteq B_{Mr}(F(x))$

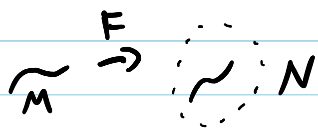
Since $A \subseteq I^m$ has measure zero, then $\forall \delta > 0$, $\exists \{B_{r_i}\}$ s.t.
$A \subseteq \bigcup_i B_{r_i}(p_i)$ with $c_m \sum_i r_i^m < \frac{\delta}{M^m}$.

$\therefore F(A) \subseteq \bigcup B_{Mr_i}(F(p_i))$

$$\sum_i \mathrm{Vol}(B_{Mr_i}) \leq M^m c_m \sum_i r_i^m < \delta.$$

$\square$

Cor: $F: M \to N$ C^1-map with $\dim M < \dim N$.
Then $F(M) \subseteq N$ has measure zero.



Proof: $M = \bigcup \mathcal{U}_i$ countable union of open charts, $\varphi_i(\mathcal{U}_i) \subseteq I^m \subseteq \mathbb{R}^m$
$\varphi_i : \tilde{\mathcal{U}}_i \to \mathbb{R}^n$, $\mathcal{U}_i \subseteq \tilde{\mathcal{U}}_i$, $I^m \subseteq \varphi_i(\tilde{\mathcal{U}}_i)$. $(\tilde{\mathcal{U}}_i, \varphi_i)$ also chart
Need: $\forall (V, \psi)$ chart for N, then $\psi(F(M) \cap V)$ has measure zero.

$\tilde{F} = \psi \circ F \circ \varphi_i^{-1}$

$\varphi_i(\mathcal{U}_i) \subseteq I^m$

$\tilde{F}: I^m \to \mathbb{R}^n$

$W_i \subseteq \tilde{F}(I^m)$

Let $W_i = \psi \circ F \circ \varphi_i^{-1}(\varphi_i(\mathcal{U}_i \cap F^{-1}(V)))$, note: $\psi(F(M) \cap V) \subseteq \bigcup_i W_i$.

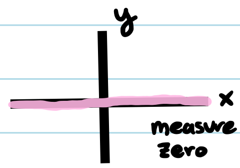
If we can show each W_i has measure zero, we're done.

Indeed: if $W_i \subseteq \bigcup_k B_i^k$, $\sum \mathrm{vol}(B_i^k) < \frac{\delta}{2^k}$, then

$$\psi(F(M) \cap V) \subseteq \bigcup_{i,k} B_i^k \text{ with } \sum_{i,k} \mathrm{vol}(B_i^k) \leq \delta \sum_k \frac{1}{2^k} = \delta.$$

So: Suffices to show: if $F: I^m \to \mathbb{R}^n$ with $m < n$ is C^1, then $F(I^m)$ has measure zero.

This is true since: $I^m \times \{0\} \subseteq \mathbb{R}^n$ has measure zero, so $F(I^m \times \{0\})$ has measure zero.
(use prev Lemma) $\square$



Sard's Thm: Let $F: M \to N$ be a smooth map.
Let $C \subseteq M$ be the set of critical points of F.
Then: $F(C) \subseteq N$ has measure zero.

Proof: We won't give the full proof. Instead, we illustrate some of the ideas by rigorously proving the following:

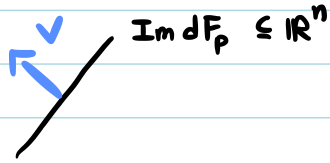
Special Case: Let $F: I^n \to \mathbb{R}^n$ be a smooth map.
$C \subseteq I^n$ critical pts
Then $F(C) \subseteq \mathbb{R}^n$ has measure zero.

Main Difficulty in general case: $F: M \to N$ with $\dim M > \dim N$.

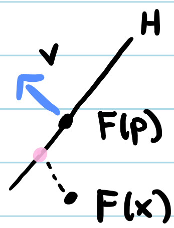
- Proved before: case $\dim M < \dim N$
- Will prove now: case $\dim M = \dim N$.

Proof of special case:

Let $c = F(p)$ be a critical pt.
$\Rightarrow$ $\mathrm{Im}\, dF_p$ has $\dim < n$.
$\exists v \in \mathbb{R}^n$ s.t. $v \perp \mathrm{Im}\, dF_p$.



Let H = plane through $F(p)$ with normal v.
Consider $F(x)$, $x \in I^n$.
$F(p) + dF_p(x-p) \in H$ $\quad dF_p(x-p) \cdot v = 0$



$$\operatorname{dist}(F(x), H) \leq |F(x) - F(p) - dF_p(x-p)|$$

(A) $\leq C|x-p|^2$ point on plane

Taylor's Thm

Taylor's Thm: Suppose $F: K \to \mathbb{R}^n$ is smooth, K compact, $\mathcal{U} \subseteq K$
$\subseteq \mathbb{R}^m$ $\mathcal{U}$ open + convex

$F = (F^1(x), \dots, F^n(x))$.

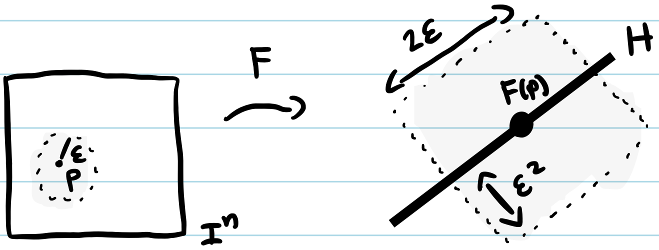
$\exists C > 0$ s.t. $\forall a \in \mathcal{U}$, $\exists R_a : \mathcal{U} \to \mathbb{R}^n$ s.t. $\forall x \in \mathcal{U}$:

$$F^i(x) = F^i(a) + \sum_p \frac{\partial F^i}{\partial x_p}(a)(x-a)_p + R_a^i(x), \quad \text{where: } |R_a^i(x)| \leq C|x-a|^2$$

Next, since F is $C^1(I^n)$ we can bound:

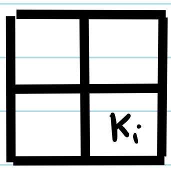
(B) $|F(x) - F(p)| \leq M|x-p| \quad \forall x, y \in I^n$.

Combine (A) + (B): If $|x-p| < \varepsilon$, then $F(x) \subseteq$ cylinder of volume $(2C\varepsilon^2)(C_n M^{n-1}\varepsilon^{n-1})$



Next: Subdivide I^n into N^n cubes K_i of edge length $1/N$

Vol = 1 = (# cubes)(vol each cube)



Pick a cube K_i. $\forall$ critical points $p \in K_i$, then:
$|x-p| \leq \sqrt{n}\, N^{-1} \quad \forall x \in K_i$

$$\{F(p) : p \in S \cap K_i\} \subseteq \text{Cylinder } A_i, \quad \operatorname{Vol}(A_i) \leq \tilde{C} N^{-2} N^{-(n-1)} = \tilde{C} N^{-n} N^{-1}$$

$$\{F(p) : p \in S \cap I^n\} \subseteq N^n \text{ cylinders } A_i$$

add up all cubes

$$\operatorname{Vol}(\cup A_i) \leq (\tilde{C} N^{-n} N^{-1}) N^n = \tilde{C} N^{-1} \to 0$$

as $N = \#\text{cubes} \to \infty$

$\square$

Application:

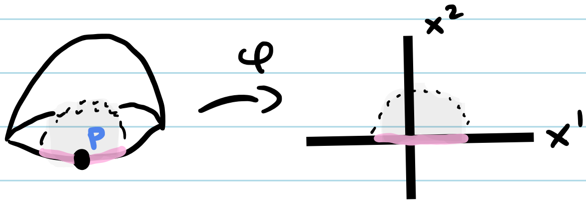
(*) **Thm:** Every smooth map $F: \overline{B_1} \to \overline{B_1}$ has a fixed pt.
($B_1 \subseteq \mathbb{R}^n$ ball radius 1 centred at 0.)

For this, first we define the notion of mfd with bdd:

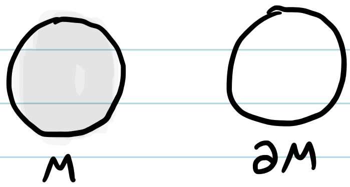
Def: An n-dimensional topological manifold with boundary is a second countable Hausdorff space s.t. every pt has a nbhd homeomorphic to either an open subset of $\mathbb{R}^n$ OR $\mathbb{H}^n$:

$$\mathbb{H}^n = \{ (x^1, \dots, x^n) \in \mathbb{R}^n : x^n \geq 0 \}.$$

Def: $\partial M = \{ p \in M \text{ s.t. } \exists \text{ chart } \varphi : \mathcal{U} \to \mathbb{H}^n \text{ with } x^n(p) = 0 \}$



ex)

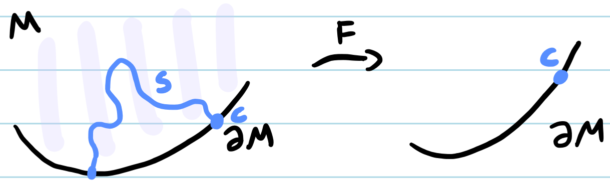


Exercise: [Lee Problem 5-23]
Suppose M mfd with bdd, N mfd, $F: M \to N$ smooth map.
Let $S = F^{-1}(c)$ where $c \in N$ regular value for: F and $F|_{\partial M}$.
Then: S is a submfd with boundary $\partial S = S \cap \partial M$.

Prop: Let M be cpt mfd with bdd.
There is no smooth $F: M \to \partial M$ with $F|_{\partial M} = id$.

Pf: Suppose such F exists.
By Sard's Thm: $\exists$ regular value $c \in \partial M$.
Since $F|_{\partial M} = id \Rightarrow c$ reg value for $F|_{\partial M}$.

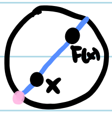
$S = F^{-1}(c) \subseteq M$ submfd of dim $= 1$, $\partial S = S \cap \partial M$.
$S \cap \partial M = \{c\}$ if $F(p) = c$, $p \in \partial M$, then $p = c$ since $F|_{\partial M} = id$.
Proof omitted → No compact 1-mfd with single boundary pt. $\square$



Proof of Thm (*): Suppose $F: \bar{B}_1 \to \bar{B}_1$ has no fixed pts.

Define: $\psi: \bar{B}_1 \to S^{n-1}$ as

$\psi(x)$ = pt in S^{n-1} closest to x on line from x to $F(x)$



$$\psi(x) = x + t \underbrace{\frac{(x - F(x))}{|x - F(x)|}}_{= u}, \quad t = -x\cdot u + \sqrt{1 - x\cdot x + (x\cdot u)^2}$$

$$(x+tu)\cdot(x+tu) = 1$$
$$t^2 + 2(u\cdot x)t + x\cdot x = 1$$

$\therefore \psi: \bar{B}_1 \to \partial B_1$ with $\psi|_{\partial B_1} = id.$ $\Rightarrow\Leftarrow$

$\square$

Whitney Embedding Thm: Every smooth n-mfd with or without boundary can be embedded into $\mathbb{R}^{2n+1}$.

Pf: Lee Theorem 6.15.
(Uses Sard's Thm)