7. Lie Groups

Def: A Lie group is a group G which is also a smooth mfd such that:

multiplication $G \times G \to G$
inversion $G \to G$

are smooth maps.

ex) $(\mathbb{R}, +)$, $(\mathbb{R}^n, +)$

ex) $S^1 = \{ e^{i\theta} : 0 \le \theta \le 2\pi \}$
$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$

ex) $GL(n, \mathbb{R}) = \{ \text{invertible } n\times n \text{ matrices} \}$
$= \{ A \in M_{n\times n}(\mathbb{R}) : \det A \neq 0 \}$

$GL(n,\mathbb{R}) \subseteq M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2} \Rightarrow GL(n,\mathbb{R})$ mfd of dim n^2.
open subset

Coords: the n^2 entries of the matrix

e.g. $X = [X^i{}_j] = \begin{pmatrix} x^1{}_1 & x^1{}_2 \\ x^2{}_1 & x^2{}_2 \end{pmatrix}$

The map $A \mapsto A^{-1}$ is smooth by Cramer's rule:

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A).$$

ex) $SL(n,\mathbb{R}) = \{ A \in GL(n,\mathbb{R}) : \det A = 1 \}$.

Showed earlier: $SL(n,\mathbb{R}) \subseteq GL(n,\mathbb{R})$ is a submfd.
$\det : GL(n,\mathbb{R}) \to \mathbb{R}$, 1 is a reg value.

ex) $O(n) = \{ A \in GL(n,\mathbb{R}) : A^T A = I \}$

$\Psi : GL(n,\mathbb{R}) \to Sym(n,\mathbb{R})$ ← identify with $\cong \mathbb{R}^{n(n+1)/2}$
$A \mapsto A^T A$ symmetric $n\times n$ matrices

Claim: I is a regular value of Ψ.

Let $A \in O(n) = \Psi^{-1}(I)$.

$$d\Psi_A(M) = \frac{d}{dt}\Big|_{t=0} (A+tM)^T(A+tM)$$
$$= M^TA + AM^T.$$

Surjective? Given $S \in \mathrm{Sym}(n, \mathbb{R})$,

$d\Psi_A(\frac{1}{2}AS) = S \Rightarrow d\Psi_A$ is surjective

$\therefore O(n) \subseteq GL(n, \mathbb{R})$ submfd
$\therefore O(n)$ Lie group.

ex) $SO(n) = \{A \in GL(n, \mathbb{R}) : A^TA = I,\ \det A = 1\}$

$= \{A \in O(n) : \det A > 0\}$. Open set in $O(n)$
$\Rightarrow SO(n) \subseteq O(n)$ is a submfd.

ex) $GL(n, \mathbb{C}) = \{A \in M_{n\times n}(\mathbb{C}) : \det A \neq 0\}$

$\subseteq \mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$ open subset

ex) $SL(n, \mathbb{C}) = \{A \in M_{n\times n}(\mathbb{C}) : \det A = 1\}$

ex) $U(n) = \{A \in M_{n\times n}(\mathbb{C}) : A^*A = I\}$, $A^* = \overline{A^T}$

Exercise: Use $\Psi : M_{n\times n}(\mathbb{C}) \to \mathrm{Herm}(n) = \{A \in M_{n\times n}(\mathbb{C}) : A^* = A\}$
$\subseteq \mathbb{R}^{n(n+1)}$

$\Psi(A) = A^*A$.

Show $U(n) = \Psi^{-1}(I)$ is a mfd.

ex) $SU(n) = \{A \in U(n) : \det A = 1\}$

$\Psi : M_{n\times n}(\mathbb{C}) \to \mathrm{Herm}(n) \times \mathbb{R}$

$\Psi(A) = (A^*A, -\frac{i}{2}(\det A - \det A^*))$

$SU(n) = \Psi^{-1}(I,0) \cap \{\det A > 0\}$

one component of $\Psi^{-1}(I,0)$ has $\det A = +1$, the other has $\det A = -1$.

$$d\Psi_A(M) = \frac{d}{dt}\Big|_{t=0} \Psi(A + tM)$$

$$= \left(M^*A + A^*M, \frac{-i}{2}\left(\det A \operatorname{Tr} A^{-1}M - \overline{\det A \operatorname{Tr} A^{-1}M}\right)\right)$$

$\frac{d}{dt} \det A(t) = (\det A) \operatorname{Tr} A^{-1}\dot{A}$.

<u>Exercise:</u> Show $d\Psi_A$ is surj. Take $(H,t) \in \operatorname{Herm}(n) \times \mathbb{R}$
$A \in SU(n)$

Verify $d\Psi_A\left(\frac{1}{2}AH + \frac{it}{n}A\right) = (H,t)$.

$\therefore$ $(I,0)$ reg value for Ψ

$\therefore$ $SU(n)$ is a mfd. Note: $SU(2)$ is connected.

<u>Prop:</u> Let G be a connected Lie group.
$W \subseteq G$ open set containing identity
Then: W generates G.

<u>Note:</u> if G = group, $S \subseteq G$, the subgroup generated by S is the smallest subgroup containing S. $H \subseteq G$ s.t. if $a,b \in H$, then: $ab \in H$
: $a^{-1} \in H$

<u>Proof:</u> Let $H \subseteq G$ be the subgroup generated by W.

1. H is open:

$$H = \bigcup_{k=1}^{\infty} W_k, \quad W_k = \{a_1 \cdots a_k : a_i \in W \cup W^{-1} \ \forall i\}$$

$W_1 = W \cup W^{-1} = W \cup \operatorname{inv}(W)$ = open

diffeo $g \mapsto g^{-1}$

$$W_k = \bigcup_{g \in W_1} g \cdot (W_{k-1}) = \text{open}$$

diffeo

2. H is closed.

$G \setminus H = \bigcup_{g \notin H} gH$ open

3. H is clopen $\Rightarrow H = G$. $\square$

<u>Def</u>: An action of a Lie group G on a manifold M is an action: $G \times M \to M$, $(g,p) \mapsto g \cdot p$ s.t. $e \cdot p = p$ (e = identity), $g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p$

which is smooth as a map of manifolds.

<u>Note</u>: $G \circlearrowright M \Rightarrow \forall g \in G$, $p \mapsto g \cdot p$ defines a diffeo $M \to M$
inverse = $p \mapsto g^{-1} \cdot p$

ex) $GL(n, \mathbb{R}) \circlearrowright \mathbb{R}^n$, $A \cdot v = Av$

ex) $O(n) \circlearrowright \mathbb{R}^n$

ex) $O(n) \circlearrowright S^{n-1}$

ex) $G \circlearrowright G$ $g \cdot h = gh$

ex) $G \circlearrowright G$, $g \cdot h = ghg^{-1}$

<u>Def</u>: $G \circlearrowright M$. Action is:

- free if: $g \cdot p = p$ for some $p \in M \Rightarrow g = e$
- transitive if: $\forall p, q \in M$, $\exists g \in G$ s.t. $g \cdot p = q$.

Orbit of $p = G \cdot p = \{g \cdot p : g \in G\}$
Stabilizer of $p := G_p = \{g \in G : g \cdot p = p\}$

Free $\Leftrightarrow$ every stabilizer is trivial.
Transitive $\Leftrightarrow$ $G \cdot p = M$ for any p.

Def: If G, H are Lie groups, then $\varphi: G \to H$ is a Lie group homomorphism if it is a smooth map s.t. $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$.

ex) $\varphi: (\mathbb{R}, +) \to (S^1, \cdot)$
$t \mapsto e^{it}$

$e^{i(t_1+t_2)} = e^{it_1} e^{it_2}$

ex) $\varphi: GL(n, \mathbb{R}) \to \mathbb{R}^*$
$A \mapsto \det A$

$\det(AB) = \det A \det B$

Def: Let G be a Lie group with identity e. Denote:
$\mathfrak{g} = \text{Lie}(G) = T_e G$.

ex) $G = GL(n, \mathbb{R})$
$\mathfrak{g} = M_{n \times n}(\mathbb{R})$

$GL(n, \mathbb{R}) \subseteq M_{n\times n}(\mathbb{R})$ open set so
$T_e GL(n, \mathbb{R}) = T_e M_{n \times n}(\mathbb{R})$

Recall matrix exponential: $\exp: M_{n\times n}(\mathbb{R}) \to GL(n, \mathbb{R})$:

$$\exp(M) = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{M^k}{k!}$$

Key properties:

(1) $\exp(0) = I$

(2) $\exp(M^T) = (\exp M)^T$

(3) $\exp(SMS^{-1}) = S \exp(M) S^{-1}$

(4) $\exp(M)\exp(-M) = I$

(5) $\det \exp(M) = \exp(\operatorname{Tr} M)$

(6) $\left.\frac{d}{dt}\right|_{t=0} \exp(tM) = M$

exp is not homomorphism

but $e^{M+N} \neq e^M e^N$!

Baker-Campbell-Hausdorff formula

(7) If $M_1 M_2 = M_2 M_1$, then
$\exp(M_1 + M_2) = \exp(M_1)\exp(M_2)$.

Useful formula:

If: $F: GL(n, \mathbb{R}) \to N$, then
$dF_e(X) = \left.\frac{d}{dt}\right|_{t=0} F(e^{tX})$, since: $\gamma(t) = e^{tX}$ solves $\gamma(0) = e$
$\dot\gamma(0) = X$.

Recall: if $\Psi : M \to N$, $c \in N$ reg value
$S = F^{-1}(c)$,
then $T_p S = \operatorname{Ker} d\Psi_p$.

ex) $G = SL(n, \mathbb{R})$
$\Psi(A) = \det A$, $SL(n,\mathbb{R}) = \Psi^{-1}(1)$.

$T_e G = \operatorname{Ker} d\Psi_e$, $d\Psi_e(M) = \operatorname{Tr} M$ (earlier)
$d\Psi_A(M) = (\det A) \operatorname{Tr} A^{-1} M$

$\Rightarrow \mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ = matrices with trace zero.

ex) $G = O(n)$
$\Psi : M_{n\times n}(\mathbb{R}) \to Sym(n, \mathbb{R})$
$A \mapsto A^T A$
$d\Psi_A(M) = M^T A + A^T M$

$\mathfrak{g} = \mathfrak{o}(n) = \{ M \in M(n, \mathbb{R}) : M^T = -M \}$

ex) $G = SO(n) = O(n) \cap \{ \det > 0 \}$
$O(n)$ and $SO(n)$ are the same near the identity.

$\mathfrak{g} = \mathfrak{so}(n) = \mathfrak{o}(n)$ = skew-sym matrices

ex) $G = SU(n)$
$\Psi : M_{n\times n}(\mathbb{C}) \to Herm \times \mathbb{R}$
$A \mapsto (A^* A, \operatorname{im} \det A)$

$d\Psi_e(M) = (M^* + M, \operatorname{im} \operatorname{Tr} M)$

$\mathfrak{g} = \mathfrak{su}(n) = \{ M \in M_{n\times n}(\mathbb{C}) : M^* = -M, \operatorname{Tr} M = 0 \}$
$\Downarrow$
TrM pure imaginary

ex) There are maps:
$\exp : \mathfrak{sl}(n, \mathbb{R}) \to SL(n, \mathbb{R})$
$\exp : \mathfrak{so}(n) \to SO(n)$
$\exp : \mathfrak{su}(n) \to SU(n)$
use properties of exp