8. Vector Fields

Def: If M is a mfd, a vector field is a smooth map $V : M \to TM$ s.t. $\pi \circ V = id$.

Coordinate representation: If V is a vector field, then over a coord chart $(\mathcal{U}, x^i)$:

$$V\big|_{\mathcal{U}} = V^i(x) \frac{\partial}{\partial x^i}, \quad V^i : \mathcal{U} \to \mathbb{R} \text{ smooth.}$$

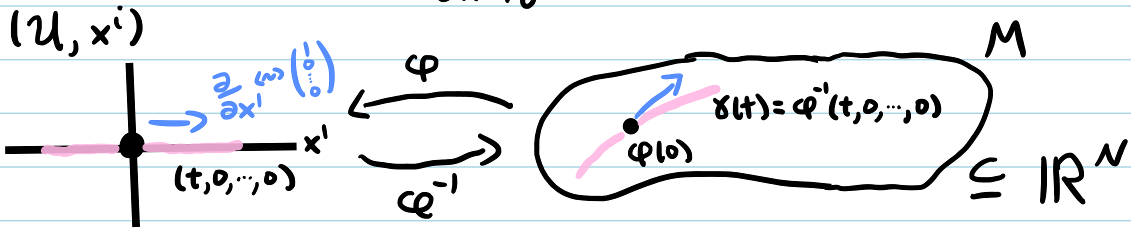
Sometimes write: for $p \in M$, $V_p = V^i(p) \frac{\partial}{\partial x^i}\Big|_p$

Think of: $V^i \frac{\partial}{\partial x^i} \leftrightsquigarrow \begin{pmatrix} V^1(x) \\ V^2(x) \\ \vdots \\ V^n(x) \end{pmatrix}$ column vector.

On overlaps $(\mathcal{U}, x^i) \cap (\tilde{\mathcal{U}}, \tilde{x}^i)$:

$$V\big|_{\mathcal{U}} = V^i(x) \frac{\partial}{\partial x^i}, \quad V\big|_{\tilde{\mathcal{U}}} = \tilde{V}^i(\tilde{x}) \frac{\partial}{\partial \tilde{x}^i} \quad \text{with: } \tilde{V}^i \overset{(*)}{=} \frac{\partial \tilde{x}^i}{\partial x^p} V^p.$$

How to visualize e.g. $\frac{\partial}{\partial x^1}\Big|_0$? Use paths: tangent vector is velocity vector.



$\frac{d\gamma}{dt}(0)$ is vector in $\mathbb{R}^N$ corresponding to "$\frac{\partial}{\partial x^1}\Big|_0$"

In general: Given a path $\gamma(t)$ on M with $\gamma(0) = p$, obtain $V \in T_pM$ via: if $(\mathcal{U}, \varphi, x^i)$ chart containing p,

$$\gamma\big|_{\mathcal{U}}(t) = \left(\gamma^1\big|_{\mathcal{U}}(t), \dots, \gamma^n\big|_{\mathcal{U}}(t)\right) := \varphi \circ \gamma(t)$$

$$V\big|_{\mathcal{U}} = \left(\frac{d}{dt}\Big|_{t=0} \gamma^i\big|_{\mathcal{U}}(t)\right) \frac{\partial}{\partial x^i}.$$

Exercise: derive $(*)$.

ex) $S^1 = \mathcal{U} \cup \tilde{\mathcal{U}}$, $\quad \mathcal{U} = \{e^{i\theta} : 0 < \theta < 2\pi\}$, $\quad \tilde{\mathcal{U}} = \{e^{i\tilde{\theta}} : -\pi < \tilde{\theta} < \pi\}$,

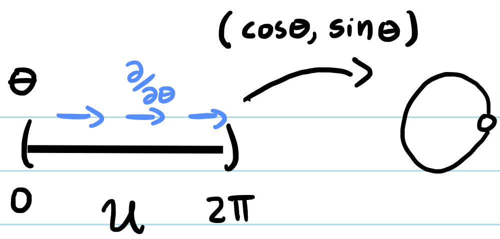
$$\tilde{\theta} = \begin{cases} \theta & \text{on } \cap \\ \theta - 2\pi & \text{on } \cup \end{cases}$$

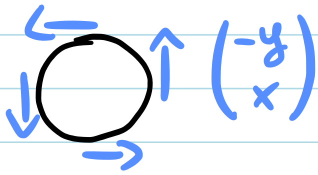
X "=" $\frac{\partial}{\partial \theta}$ is a well-defined vector field: $\tilde{X}^{\tilde{\theta}} \overset{?}{=} \frac{\partial \tilde{\theta}}{\partial \theta} X^{\theta}$.

Technically:

$$X\big|_{\mathcal{U}} = X^\theta \frac{\partial}{\partial\theta} = 1\frac{\partial}{\partial\theta} \qquad 1 = \frac{\partial\tilde{\theta}}{\partial\theta}\, 1$$

$$X\big|_{\tilde{\mathcal{U}}} = \tilde{X}^{\tilde{\theta}} \frac{\partial}{\partial\tilde{\theta}} = 1\frac{\partial}{\partial\tilde{\theta}}$$





notation:
$\gamma : (-\varepsilon,\varepsilon) \to M$
$(\mathcal{U},\varphi)$ chart
$\gamma\big|_{\mathcal{U}} = \varphi\circ\gamma$

Show: $\partial/\partial\theta \leftrightsquigarrow \begin{pmatrix}-y\\x\end{pmatrix}$ in $\mathbb{R}^2$.

Let $X\big|_{\mathcal{U}}(\theta_0) = \frac{\partial}{\partial\theta}\Big|_{\theta_0} \in T_{\theta_0}S^1$ with $0 < \theta_0 < 2\pi$.

Consider: $\gamma(t) = (\cos(t+\theta_0), \sin(t+\theta_0)) = (x(t), y(t))$

$\gamma(0) = e^{i\theta_0}$.

$\gamma\big|_{\mathcal{U}}(t) = \theta\circ\gamma(t) = t+\theta_0, \quad X\big|_{\mathcal{U}}(\theta_0) = \left(\frac{d}{dt}\Big|_{t=0}\gamma\big|_{\mathcal{U}}\right)\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\theta}$.

Corresponding vector in $\mathbb{R}^2$: $\frac{d\gamma}{dt}(0) = \begin{pmatrix}-\sin\theta_0\\ \cos\theta_0\end{pmatrix} = \begin{pmatrix}-y\\x\end{pmatrix}$.

ex) $S^2 =$  $\cup$  stereographic coords

$\mathcal{U}$ (u,v) $\quad$ $\tilde{\mathcal{U}}$ $(\tilde{u},\tilde{v})$

$$(\tilde{u},\tilde{v}) = \frac{1}{u^2+v^2}(u,v).$$

Suppose W is a vector field on S^2 s.t.

$W\big|_{\mathcal{U}} = \frac{\partial}{\partial u}$. Show $W = 0$ at the North pole.

$$\frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)} = \begin{pmatrix}\frac{\partial\tilde{u}}{\partial u} & \frac{\partial\tilde{u}}{\partial v}\\ \frac{\partial\tilde{v}}{\partial u} & \frac{\partial\tilde{v}}{\partial v}\end{pmatrix} = \begin{pmatrix}\tilde{v}^2-\tilde{u}^2 & -2\tilde{u}\tilde{v}\\ -2\tilde{u}\tilde{v} & \tilde{u}^2-\tilde{v}^2\end{pmatrix}$$

exercise

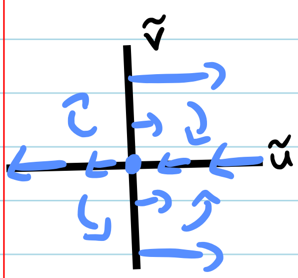
$(x^1,x^2) = (u,v)$

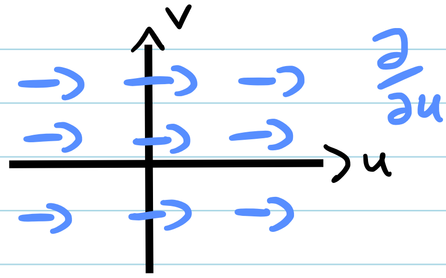
$$\tilde{W}^i = \frac{\partial\tilde{x}^i}{\partial x^p}W^p$$

$$\begin{pmatrix}\tilde{W}^1\\ \tilde{W}^2\end{pmatrix} = \left(\frac{\partial\tilde{x}^i}{\partial x^j}\right)_{2\times2}\begin{pmatrix}W^1\\W^2\end{pmatrix} \Rightarrow \begin{pmatrix}\tilde{W}^1\\ \tilde{W}^2\end{pmatrix} = \begin{pmatrix}\tilde{v}^2-\tilde{u}^2 & -2\tilde{u}\tilde{v}\\ -2\tilde{u}\tilde{v} & \tilde{u}^2-\tilde{v}^2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$W\big|_{\tilde{u}} = (\tilde{v}^2 - \tilde{u}^2)\frac{\partial}{\partial \tilde{u}} - 2\tilde{u}\tilde{v}\frac{\partial}{\partial \tilde{v}}.$$

$W\big|_{(0,0)} = 0$ at $(\tilde{u}, \tilde{v}) = (0,0)$.





$(\tilde{v}^2 - \tilde{u}^2)\, \partial/\partial \tilde{u}$

$-2\tilde{u}\tilde{v}\, \frac{\partial}{\partial \tilde{v}}$

Def: M is parallelizable if it admits a smooth global frame.

Global frame: vector fields $\{X_1, \dots, X_n\}$ s.t. $\{X_1|_p, \dots, X_n|_p\}$ are a basis for T_pM $\forall p \in M$.

ex) S^1 is parallelizable: $\frac{\partial}{\partial \theta}$.

ex) S^2 is not parallizable.
Famous Thm: every vector field must vanish somewhere.

Def: $F: M \to N$ diffeo.
X vector field on M
$F_* X$ vector field on N defined by (push forward)
$(F_* X)_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)})$.

Recall: $dF_p : T_pM \to T_{F(p)}N$

Coords: $F_* X\big|_V = \frac{\partial F^i}{\partial x^p}(F^{-1}(y))\, X^p(F^{-1}(y))\, \frac{\partial}{\partial y^i}$

$(\mathcal{U}, x^i)$ on M
(V, y^i) on N
$F\big|_{\mathcal{U}} = (F^1(x), \dots, F^n(x))$

Def: Let M be mfd

: V, W vector fields

Lie bracket:

$$[V,W] = \left(V^i \frac{\partial W^k}{\partial x^i} - W^i \frac{\partial V^k}{\partial x^i} \right) \frac{\partial}{\partial x^k}, \qquad V = V^i \frac{\partial}{\partial x^i}$$

$$W = W^i \frac{\partial}{\partial x^i}$$

Exercise: check $[V,W]$ is a well-defined vector field:

$$\widetilde{[V,W]}^i = \frac{\partial \tilde{x}^i}{\partial x^p} [V,W]^p.$$

Properties:

(1) $[a_1 V_1 + b_1 V_2, W] = a_1 [V_1, W] + a_2 [V_2, W]$

(2) $[V,W] = -[W,V]$

(3) $[V,[W,X]] + [W,[X,V]] + [X,[V,W]] = 0$ Jacobi identity

Prop: Let $F: M \to N$ be a diffeo.

Then $F_*[V,W] = [F_* V, F_* W]$.

Pf: Omitted.

Back to Lie groups

Let G be a Lie group. Let $g \in G$. Let $e \in G$ denote the identity.

$L_g : G \to G$

$L_g(h) = gh$ is a diffeo. $L_g^{-1} = L_{g^{-1}}$.

Given $X \in T_e G$, can define: $X_g = (L_g)_* X$. $(L_g)_* (L_{g^{-1}})_* = id$

If $X \neq 0 \Rightarrow$ obtain non-vanishing vector field since $\ker(L_g)_* = \{0\}$.

Given basis: $e_1\big|_e, \dots, e_n\big|_e \in T_e G$,

obtain global frame: $\underset{\substack{\| \\ e_1}}{(L_g)_* e_1\big|_e}, \dots, \underset{\substack{\| \\ e_n}}{(L_g)_* e_n\big|_e}$

$\therefore$ G is parallelizable.

Note: $TG \cong G \times \mathbb{R}^n$ trivial tangent bundle

$$F: G \times \mathbb{R}^n \to TG$$
$$F\left(g, \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}\right) = v^i e_i\big|_g \qquad \text{diffeo.}$$

Def: A vector field X on G is left-invariant if:
$(L_g)_* X_h = X_{gh}$. (Same as $X_g = (L_g)_* X_e$)

Correspondence: $T_e G \rightsquigarrow$ { left invariant vector fields on G }
$X \in T_e G$ $\qquad X_g = (L_g)_* X$

Note: If V, W left-invariant
then $[V,W]$ is left-invariant.

Indeed: $(L_g)_* [V,W] = [(L_g)_* V, (L_g)_* W] = [V, W]$.

$\therefore$ $[\cdot,\cdot]$ equips $\mathfrak{g} = T_e G = \{ \text{left-inv VF} \}$ with a bracket.

$X_e, Y_e \in \mathfrak{g} \rightsquigarrow X, Y$ left-inv VF $\rightsquigarrow [X,Y]$ left inv VF

$$\Downarrow$$
$$[X,Y]\big|_e \in \mathfrak{g}$$

$(\mathfrak{g}, [\cdot,\cdot])$ is the Lie algebra of the Lie group G.

Main Example: $G = GL(n, \mathbb{R})$
$\mathfrak{g} = M_{n \times n}(\mathbb{R})$

$[A, B]$ = matrix commutator.

Indeed: $A \in T_e G$, $\quad A = [A^i{}_j]_{n \times n}$
$$A = A^i{}_j \frac{\partial}{\partial x^i{}_j}$$

Left-inv VF?

$$(L_g)_* A = \frac{d}{dt}\Big|_{t=0} L_g \circ \gamma(t), \qquad \gamma(t) \text{ path in } GL(n,\mathbb{R})$$
$$\gamma(0) = g, \quad \dot\gamma(0) = A$$
$$= gA$$
$$= x^i{}_p(g) A^p{}_j \frac{\partial}{\partial x^i{}_j} \qquad x^i{}_p(g) = g^i{}_p = (ip)^{th} \text{ entry of } g$$

Bracket? $A, B \in T_e G$

$$[A,B] = \left(x^i{}_p A^p{}_j \frac{\partial}{\partial x^i{}_j} \left(x^k{}_\ell B^\ell{}_m \right) - x^i{}_p B^p{}_j \frac{\partial}{\partial x^i{}_j} \left(x^k{}_\ell A^\ell{}_m \right) \right) \frac{\partial}{\partial x^k{}_m}$$

$$= \left(x^k{}_p A^p{}_j B^j{}_m - x^k{}_p B^p{}_j A^j{}_m \right) \frac{\partial}{\partial x^k{}_m}$$

$$[A,B]\big|_e = \left(\delta^k{}_p A^p{}_j B^j{}_m - \delta^k{}_p B^p{}_j A^j{}_m \right) \frac{\partial}{\partial x^k{}_m}$$

$$= [A,B]^k{}_m \frac{\partial}{\partial x^k{}_m}$$

matrix commutator