9. Flows of Vector Fields

Def: M be a smooth mfd. A global flow on M is a smooth map

$$\Theta : \mathbb{R} \times M \to M$$

$$\Theta_t(x) = \Theta(t,x)$$

s.t.

$$\Theta_t \circ \Theta_s = \Theta_{t+s}, \quad \Theta_0 = id_M.$$

Note: For each $t \in \mathbb{R}$,

$\Theta_t : M \to M$ is a diffeo. $\quad \Theta_t \, \Theta_{-t} = id$

ex) $\Theta_t : \mathbb{R}^2 \to \mathbb{R}^2$

$\Theta_t(x,y) = (x+t, y)$

ex) $\Theta_t : GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$

$\Theta_t(A) = \exp(tM)A$, for some $M \in M_{n\times n}(\mathbb{R})$

Prop: Let Θ_t be a global flow on M. Then $X = \left.\frac{d}{dt}\right|_{t=0} \Theta_t$

is a vector field.

Pf: $\forall p \in M$, $\gamma(t) = \Theta_t(p)$ is a path with $\gamma(0) = p$.

$\therefore \dot{\gamma}(0) \in T_pM$

$\therefore \left.\frac{d}{dt}\right|_{t=0} \Theta_t(p) \in T_pM \quad \forall p \in M$. Is X smooth? Look in coords:

In a coordinate chart $(\mathcal{U}, x^i)$:

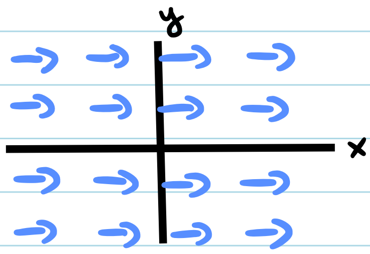
$$\Theta_t\big|_{\mathcal{U}} = (\Theta^1_t(x), \dots, \Theta^n_t(x))$$

$$X\big|_{\mathcal{U}} = \left.\frac{d}{dt}\right|_{t=0} \underbrace{\Theta^i_t(x)}_{\text{smooth}} \frac{\partial}{\partial x^i} = X^i \frac{\partial}{\partial x^i} \quad \square$$

ex) $\Theta_t : \mathbb{R}^2 \to \mathbb{R}^2$

$\Theta_t(x,y) = (x+t, y)$

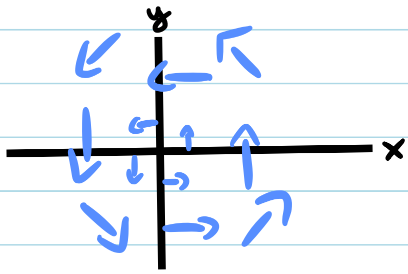
$$X = \left.\frac{d}{dt}\right|_{t=0} \Theta_t = \frac{\partial}{\partial x}$$



ex) $\Theta_t : \mathbb{R}^2 \to \mathbb{R}^2$

$\Theta_t(x,y) = (x\cos t - y\sin t,\ x\sin t + y\cos t)$

$$X = \frac{d}{dt}\Big|_{t=0} \Theta_t = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$



Thm: Let M be a compact mfd.
Let X be a smooth vector field.

$\exists!$ global flow $\Theta_t : M \to M$ s.t. $\frac{d}{dt}\Big|_{t=0} \Theta_t = X$.

Pf: [Lee Corollary 9.17]. $\square$

But if M is not compact, the flow may not exist $\forall t$.

ex) $X = x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}$.

$\Theta_t(x) = x(1-tx)^{-1}$ check: $\Theta_t \circ \Theta_s = \Theta_{t+s}$

$\frac{d}{dt}\Theta_t(x) = -\frac{x(-x)}{(1-tx)^2}$

Only defined on $\{1 - tx > 0\} \subseteq \mathbb{R} \times \mathbb{R}$.

Thm: Let X be a smooth vector field on a smooth mfd M.
$\exists$ open set $\mathcal{U} \subseteq \mathbb{R} \times M$ containing $\{0\} \times M$ s.t.

(*) $\Theta : \mathcal{U} \to M$ is a local flow with $X = \frac{d}{dt}\Big|_{t=0} \Theta_t$.

Local flow: $\Theta_t \circ \Theta_s = \Theta_{t+s}$, $\Theta_0 = id_M$ $\quad$ $\Theta_t(x) = \Theta(t,x)$, $(t,x) \in \mathcal{U}$

If $\tilde{\Theta} : \tilde{\mathcal{U}} \to M$ satisfies (*), then $\tilde{\Theta} = \Theta$ on $\mathcal{U} \cap \tilde{\mathcal{U}}$.

Pf: [Lee Theorem 9.12]. $\square$

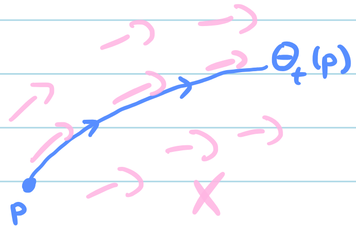
Prop: Let θ_t be a global flow with $X = \frac{d}{dt}\Big|_{t=0} \theta_t$.

Then: $\frac{d}{dt}\theta_t(p) = X_{\theta_t(p)}$.

Pf: $\frac{d}{dt}\Big|_{t=t_0} \theta_t(p) = \frac{d}{d\tilde{t}}\Big|_{\tilde{t}=0} \theta_{\tilde{t}+t_0}(p)$ $\quad \tilde{t} = t - t_0$

$$= \frac{d}{d\tilde{t}}\Big|_{\tilde{t}=0} \theta_{\tilde{t}}(\theta_{t_0}(p))$$

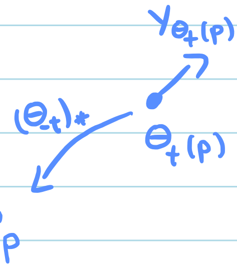
$$= X_{\theta_{t_0}(p)}. \quad \square$$



Def: Let X, Y be vector fields.

Lie derivative of Y wrt X $= (L_X Y)_p := \frac{d}{dt}\Big|_{t=0} (\theta_{-t})_* Y_{\theta_t(p)}$

where θ_t is the flow of X.



Prop: $L_X Y = [X, Y]$.

Pf: $\frac{d}{dt}\Big|_{t=0} (\theta_{-t})_* Y_{\theta_t(p)} = \frac{d}{dt}\Big|_{t=0} \frac{\partial \theta_{-t}^{\,k}(x)}{\partial x^i} Y^i(\theta_t(x)) \frac{\partial}{\partial x^k}$ $\quad \theta_0(x) = x$

$$= -\partial_i X^k Y^i \partial_k + \delta_i^{\,k} \partial_\ell Y^i X^\ell \partial_k$$

$$= -Y^i \partial_i X^k \partial_k + X^\ell \partial_\ell Y^k \partial_k$$

$$= [X, Y]^k \partial_k. \quad \square$$

Prop: $[X, Y] = 0 \iff (\theta_t)_* Y_p = Y_{\theta_t(p)}$, θ_t flow of X.

Pf: $(\Rightarrow)$ Suppose $[X, Y] = 0$, let θ_t be flow of X.

$\frac{d}{dt}\Big|_{t=t_0} (\theta_{-t})_* Y_{\theta_t(p)} = \frac{d}{d\tilde{t}}\Big|_{\tilde{t}=0} (\theta_{-t_0})_* (\theta_{-\tilde{t}})_* Y_{\theta_{\tilde{t}}(\theta_{t_0}(p))}$ $\quad \tilde{t} = t - t_0$

$$= (\Theta_{-t_0})_* \, [X,Y]_{\Theta_{t_0}(p)}$$

$$\left.\frac{d}{dt}\right|_{t=0} (\Theta_{-t})_* Y_{\Theta_t(p)} = [X,Y]_p$$

$$= 0$$

$$\therefore \frac{d}{dt}\left((\Theta_{-t})_* Y_{\Theta_t(p)} \right) \equiv 0$$

$$\Rightarrow (\Theta_{-t})_* Y_{\Theta_t(p)} = (\Theta_0)_* Y_{\Theta_0(p)} = Y_p \qquad \text{const in time}$$

$$\Rightarrow Y_{\Theta_t(p)} = (\Theta_t)_* Y_p \qquad (\Theta_t)_* (\Theta_{-t})_* = \mathrm{id}$$

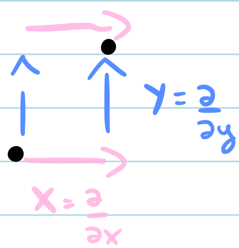
($\Leftarrow$) Run argument in reverse. Omitted. $\square$

Thm: Let M be a compact mfd

Let Θ_t, ψ_s be global flows with: Θ_t flow of X

ψ_s flow of Y.

$$[X,Y] = 0 \iff \Theta_t \circ \psi_s = \psi_s \circ \Theta_t$$



$[X,Y] = 0$

$\Theta_t \circ \psi_s = \psi_s \circ \Theta_t$

Pf: ($\Rightarrow$) Suppose $[X,Y]=0$.

Fix s. Let $y(t,p) = \Theta_t \circ \psi_s(p)$

$\tilde{y}(t,p) = \psi_s \circ \Theta_t(p)$.

$$\frac{d}{dt} y(t,p) = \frac{d}{dt} \Theta_t(\psi_s(p)) = X_{\Theta_t \circ \psi_s(p)} \qquad \frac{d}{dt}\Theta_t(p) = X_{\Theta_t(p)}$$

$$\frac{d}{dt} \tilde{y}(t,p) = \frac{d}{dt} \psi_s \circ \Theta_t(p) = (\psi_s)_* X_{\Theta_t(p)}$$

$$= X_{\psi_s \circ \Theta_t(p)}$$

$[X,Y]=0$ implies (prev prop) $(\psi_s)_* X_p = X_{\psi_s(p)}$

In a coord chart ($|s-t|<\varepsilon$):

$$\therefore \frac{d}{dt} |\tilde{y}-y|^2 = 2\left\langle \tilde{y}-y, \frac{d}{dt}(\tilde{y}-y) \right\rangle$$

$$\leq 2|\tilde{y}-y| \, |X_{\underbrace{\psi_s \circ \Theta_t}_{\tilde{y}}} - X_{\underbrace{\Theta_t \circ \psi_s}_{y}}|$$

$$X_y = \left[X^i(y(t,p)) \right]$$

$$\leq c|\tilde{y}-y|^2$$

X is Lipschitz

$$\therefore \frac{d}{dt}\left(e^{-ct} |\tilde{y}-y|^2 \right) \leq e^{-ct}(-c+c)|\tilde{y}-y|^2 = 0$$

$\therefore \; e^{-ct} |\tilde{y} - y|^2 \equiv \text{const}$

$\therefore \; |\tilde{y} - y| \equiv 0$ since $\tilde{y}(0,p) = y(0,p) = \psi_s(p)$.

($\Leftarrow$) omitted.

$\square$

<u>Thm</u>: Let M be a smooth mfd

$V_1, \dots, V_K$ linearly independent vector fields with $[V_i, V_j] = 0 \;\; \forall i,j$.

Then: $\forall p \in M$, $\exists$ chart $(\mathcal{U}, y^i)$ s.t. $V_i = \dfrac{\partial}{\partial y^i} \quad \forall i = 1, \dots, k$.

<u>Pf</u>: [Lee Thm 9.46].