

Problem Set 1: Manifolds and Maps

Problem 1. (Stereographic Projection) Let $S^2 \subset \mathbb{R}^3$ be the 2-sphere.

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Let $N = (0, 0, 1)$ and $S = (0, 0, -1)$ denote the north and south poles. Define

$$U = S^2 \setminus \{N\}, \quad \tilde{U} = S^2 \setminus \{S\}$$

and consider the open cover

$$S^2 = U \cup \tilde{U}.$$

Define local coordinates by

$$\varphi : U \rightarrow \mathbb{R}^2, \quad \varphi(x, y, z) = (u, v),$$

where $(u, v, 0)$ is the unique point where the line through (x, y, z) and N intersects the $\{z = 0\}$ plane.

We can also define

$$\tilde{\varphi} : \tilde{U} \rightarrow \mathbb{R}^2, \quad \tilde{\varphi}(x, y, z) = (\tilde{u}, \tilde{v}),$$

where $(\tilde{u}, \tilde{v}, 0)$ is the point of intersection of the plane $\{z = 0\}$ and the line through (x, y, z) and S .

Give explicit formulas for φ and $\tilde{\varphi}$, and compute the coordinate change $\tilde{\varphi} \circ \varphi^{-1}$.

Problem 2. Let

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$$

where $\sim$ is the equivalence relation

$$(x_0, \dots, x_n) \sim (y_0, \dots, y_n)$$

if

$$(x_0, \dots, x_n) = (\lambda y_0, \dots, \lambda y_n)$$

for some $\lambda \in \mathbb{R} \setminus \{0\}$. The equivalence class of a point $x = (x_0, \dots, x_n)$ will be denoted

$$[x_0, \dots, x_n],$$

with square brackets. We can cover $\mathbb{RP}^n$ by the open sets

$$U_i = \{[x_0, \dots, x_n] : x_i \neq 0\}.$$

Define coordinates $\varphi_i : U_i \rightarrow \mathbb{R}^n$ by

$$\varphi_i[x_0, \dots, x_n] = \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right) = (u^1, \dots, u^n).$$

We then have two different coordinates charts on $U_0 \cap U_1$: let us denote these by (U_0, u) and $(U_1, \tilde{u})$. Compute the coordinate change formula $\tilde{u}^i = f^i(u)$ on $U_0 \cap U_1$.

Problem 3. Let $G(2, 4)$ be the set of all 2-dimensional subspaces of $\mathbb{R}^4$. A point V in $G(2, 4)$ can be represented by a matrix

$$\begin{bmatrix} v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{bmatrix},$$

where $v = (v_1, v_2, v_3, v_4)$ and $w = (w_1, w_2, w_3, w_4)$ span $V \subset \mathbb{R}^4$. If $V \in G(2, 4)$ is represented by a 2×4 matrix A , we will write $V = [A]$.

(a) Suppose a 2×4 matrix A can be brought to the form

$$\begin{bmatrix} 1 & 0 & x_1 & x_3 \\ 0 & 1 & x_2 & x_4 \end{bmatrix}$$

by row operations. Show that such a matrix is unique, i.e. the x_1, x_2, x_3, x_4 are uniquely determined.

(b) Show that if A can be brought to the form above by row operations, then any other matrix whose coefficients are close to the coefficients of A can also be brought to that form.

(c) Show that any 2×4 matrix of rank 2 can be brought to one of the following matrices

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}, \begin{bmatrix} 1 & * & 0 & * \\ 0 & * & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & 0 \\ 0 & * & * & 1 \end{bmatrix},$$

$$\begin{bmatrix} * & 1 & 0 & * \\ * & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} * & 1 & * & 0 \\ * & 0 & * & 1 \end{bmatrix}, \begin{bmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix},$$

by row operations. Conclude that any point $V \in G(2, 4)$ can be represented by $V = [A]$, where A has the form of one of the matrices above.

(d) Cover $G(2, 4)$ by the sets

$$U_1 = \left\{ V \in G(2, 4) : V = \left[\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix} \right] \right\},$$

$$U_2 = \left\{ V \in G(2, 4) : V = \left[\begin{bmatrix} 1 & * & 0 & * \\ 0 & * & 1 & * \end{bmatrix} \right] \right\},$$

and so on for $U_3, \dots, U_6$. By the argument in (b), these sets are open. Define coordinates $\varphi_1 : U_1 \rightarrow \mathbb{R}^4$ by

$$\begin{aligned}\varphi_1 \left(\begin{bmatrix} 1 & 0 & x_1 & x_3 \\ 0 & 1 & x_2 & x_4 \end{bmatrix} \right) &= (x_1, x_2, x_3, x_4), \\ \varphi_2 \left(\begin{bmatrix} 1 & y_1 & 0 & y_3 \\ 0 & y_2 & 1 & y_4 \end{bmatrix} \right) &= (y_1, y_2, y_3, y_4),\end{aligned}$$

and similarly for the other (U_i, φ_i) . Compute the coordinate change $\varphi_1 \circ \varphi_2^{-1}$.

A similar argument shows that all $\varphi_i \circ \varphi_j^{-1}$ are smooth functions, and hence $G(2, 4)$ is given the structure of a smooth manifold. In fact, this argument can be generalized to the Grassmannian $G(k, n)$, which is the space of k -planes in $\mathbb{R}^n$.

Problem 4. Let

$$S^1 = \{e^{i\theta} : \theta \in [0, 2\pi]\}.$$

Cover S^1 by the following two open sets:

$$\begin{aligned}U &= \{e^{i\theta} : \theta \in (0, 2\pi)\} \\ \tilde{U} &= \{e^{i\tilde{\theta}} : \tilde{\theta} \in (-\pi, \pi)\}.\end{aligned}$$

Use the cover $S^1 = U \cup \tilde{U}$ to equip the tangent bundle TS^1 with coordinates. Compute the change of coordinates on TS^1 , and conclude that TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$.

Problem 5. Equip S^2 with stereographic coordinates:

$$S^2 = U \cup \tilde{U}, \quad U = S^2 \setminus \{N\}, \quad \tilde{U} = S^2 \setminus \{S\}$$

Let (u^1, u^2) denote stereographic coordinates on U , let $(\tilde{u}^1, \tilde{u}^2)$ denote stereographic coordinates on $\tilde{U}$, and denote the coordinate change by

$$(\tilde{u}^1, \tilde{u}^2) = (f^1(u), f^2(u)).$$

(a) Compute the change of coordinates on the tangent bundle TS^2 . Recall that this is of the form:

$$(\tilde{u}^1, \tilde{u}^2, \tilde{q}^1, \tilde{q}^2) = \left(f^1(u), f^2(u), \frac{\partial \tilde{u}^1}{\partial u^i} q^i, \frac{\partial \tilde{u}^2}{\partial u^i} q^i \right). \quad (1)$$

(b) We can use these calculations to show that every vector field on S^2 must vanish somewhere. Recall that a vector field is a smooth map

$$V : S^2 \rightarrow TS^2$$

satisfying $\pi \circ V = id$, where $\pi : TS^2 \rightarrow S^2$ is the bundle projection locally given by $(u, q) \mapsto u$. In the local coordinates above, a vector field creates a pair of maps

$$\begin{bmatrix} q^1(u) \\ q^2(u) \end{bmatrix} : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad \begin{bmatrix} \tilde{q}^1(\tilde{u}) \\ \tilde{q}^2(\tilde{u}) \end{bmatrix} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad (2)$$

satisfying the change of coordinates relation

$$\tilde{q}^i = \frac{\partial \tilde{u}^i}{\partial u^k} q^k.$$

Show that it is impossible for both functions in (2) to be nowhere zero. The formulas simplify in complex variables. Let

$$z = u^1 + iu^2, \quad \hat{z} = \tilde{u}^1 + i\tilde{u}^2, \quad Q = q^1 + iq^2, \quad \hat{Q} = \tilde{q}^1 + i\tilde{q}^2.$$

The change of coordinates relations (1) on the overlap $\mathbb{C}^*$ are then

$$\hat{z} = \frac{1}{\bar{z}}, \quad \hat{Q} = -\frac{1}{\bar{z}^2} \bar{Q}.$$

Assuming both $Q(z)$ and $\hat{Q}(\hat{z})$ are both nowhere zero smooth functions on $\mathbb{C}$, one approach to a contradiction is to compare the winding number of the image curve $Q(C_r)$, where $C_r = \{|z| = r\}$, for both small and large radius $r > 0$.

Problem 6. Show that the map $F : S^2 \rightarrow \mathbb{CP}^1$ given by

$$F(x, y, z) = \begin{cases} [x + iy, 1 - z] & \text{if } z \neq 1, \\ [1, 0] & \text{if } (x, y, z) = (0, 0, 1) \end{cases}$$

is a diffeomorphism.

Problem 7. Consider the map $F : S^2 \rightarrow \mathbb{RP}^2$ given by

$$F(x, y, z) = [x, y, z].$$

Show that F is smooth. Is F surjective? Is F injective?

Problem 8. Show that $F : S^1 \rightarrow \mathbb{RP}^1$ given by

$$F(e^{i\theta}) = [\cos(\theta/2), \sin(\theta/2)]$$

is a diffeomorphism.
