

Problem Set 1: Manifolds and Maps

Problem 1. (Stereographic Projection) Let $S^2 \subset \mathbb{R}^3$ be the 2-sphere.

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Let $N = (0, 0, 1)$ and $S = (0, 0, -1)$ denote the north and south poles. Define

$$U = S^2 \setminus \{N\}, \quad \tilde{U} = S^2 \setminus \{S\}$$

and consider the open cover

$$S^2 = U \cup \tilde{U}.$$

Define local coordinates by

$$\varphi : U \rightarrow \mathbb{R}^2, \quad \varphi(x, y, z) = (u, v),$$

where $(u, v, 0)$ is the unique point where the line through (x, y, z) and N intersects the $\{z = 0\}$ plane.

We can also define

$$\tilde{\varphi} : \tilde{U} \rightarrow \mathbb{R}^2, \quad \tilde{\varphi}(x, y, z) = (\tilde{u}, \tilde{v}),$$

where $(\tilde{u}, \tilde{v}, 0)$ is the point of intersection of the plane $\{z = 0\}$ and the line through (x, y, z) and S .

Give explicit formulas for φ and $\tilde{\varphi}$, and compute the coordinate change $\tilde{\varphi} \circ \varphi^{-1}$.

Solution. We start with the explicit formula for $\varphi(x, y, z)$. The line through (x, y, z) and N is parametrized by

$$r(t) = t(x, y, z) + (1 - t)(0, 0, 1) \quad t \in \mathbb{R}.$$

The intersection of $\{z = 0\}$ and $r(t)$ occurs when $tz + 1 - t = 0$ which is at time

$$t = \frac{1}{1 - z}.$$

Therefore the line encounters the plane at the point

$$r\left(\frac{1}{1 - z}\right) = \left(\frac{x}{1 - z}, \frac{y}{1 - z}, 0\right),$$

and so

$$\varphi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right) = (u, v).$$

We must now invert φ . The equation above is

$$x = u(1-z), \quad y = v(1-z), \tag{1}$$

and we need to solve for x, y, z . We will solve for z first and then substitute into (1). For this, we start by substituting (1) into the sphere equation $x^2 + y^2 + z^2 = 1$, and get

$$u^2(1-z)^2 + v^2(1-z)^2 + z^2 = 1$$

which simplifies to

$$(u^2 + v^2)(1-z)^2 = (1-z)(1+z) \implies (u^2 + v^2)(1-z) = (1+z)$$

since $z \neq 1$. We can now solve for z .

$$z = \frac{u^2 + v^2 - 1}{1 + u^2 + v^2}.$$

Substituting this into (1) gives x and y . The result is

$$\varphi^{-1}(u, v) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{u^2 + v^2 - 1}{1 + u^2 + v^2} \right) = (x, y, z). \tag{2}$$

We can redo this calculation to derive coordinates using the south pole S . We omit the details and simply record the answer:

$$\tilde{\varphi}(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z} \right) = (\tilde{u}, \tilde{v}),$$

and

$$\tilde{\varphi}^{-1}(\tilde{u}, \tilde{v}) = \left(\frac{2\tilde{u}}{1 + \tilde{u}^2 + \tilde{v}^2}, \frac{2\tilde{v}}{1 + \tilde{u}^2 + \tilde{v}^2}, \frac{1 - \tilde{u}^2 - \tilde{v}^2}{1 + \tilde{u}^2 + \tilde{v}^2} \right) = (x, y, z). \tag{3}$$

The change of coordinates

$$\tilde{\varphi} \circ \varphi^{-1} : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}^2 \setminus \{0\}$$

can now be directly computed using these explicit formulas. The result is

$$(\tilde{u}, \tilde{v}) = \left(\frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2} \right).$$

□

Problem 2. Let

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$$

where $\sim$ is the equivalence relation

$$(x_0, \dots, x_n) \sim (y_0, \dots, y_n)$$

if

$$(x_0, \dots, x_n) = (\lambda y_0, \dots, \lambda y_n)$$

for some $\lambda \in \mathbb{R} \setminus \{0\}$. The equivalence class of a point $x = (x_0, \dots, x_n)$ will be denoted

$$[x_0, \dots, x_n],$$

with square brackets. We can cover $\mathbb{RP}^n$ by the open sets

$$U_i = \{[x_0, \dots, x_n] : x_i \neq 0\}.$$

Define coordinates $\varphi_i : U_i \rightarrow \mathbb{R}^n$ by

$$\varphi_i[x_0, \dots, x_n] = \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right) = (u^1, \dots, u^n).$$

We then have two different coordinates charts on $U_0 \cap U_1$: let us denote these by (U_0, u) and $(U_1, \tilde{u})$. Compute the coordinate change formula $\tilde{u}^i = f^i(u)$ on $U_0 \cap U_1$.

Solution. The question asks to compute $\varphi_1 \circ \varphi_0^{-1}$. The inverse is

$$\varphi_0^{-1}(u) = [1, u^1, u^2, \dots, u^n].$$

Therefore

$$\varphi_1 \circ \varphi_0^{-1}(u) = \left(\frac{1}{u^1}, \frac{u^2}{u^1}, \dots, \frac{u^n}{u^1} \right).$$

This can also be written as

$$\tilde{u}^1 = \frac{1}{u^1}, \quad \tilde{u}^j = \frac{u^j}{u^1}, \quad 2 \leq j \leq n.$$

□

Problem 3. Let $G(2, 4)$ be the set of all 2-dimensional subspaces of $\mathbb{R}^4$. A point V in $G(2, 4)$ can be represented by a matrix

$$\begin{bmatrix} v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{bmatrix},$$

where $v = (v_1, v_2, v_3, v_4)$ and $w = (w_1, w_2, w_3, w_4)$ span $V \subset \mathbb{R}^4$. If $V \in G(2, 4)$ is represented by a 2×4 matrix A , we will write $V = [A]$.

(a) Suppose a 2×4 matrix A can be brought to the form

$$\begin{bmatrix} 1 & 0 & x_1 & x_3 \\ 0 & 1 & x_2 & x_4 \end{bmatrix}$$

by row operations. Show that such a matrix is unique, i.e. the x_1, x_2, x_3, x_4 are uniquely determined.

(b) Show that if A can be brought to the form above by row operations, then any other matrix whose coefficients are close to the coefficients of A can also be brought to that form.

(c) Show that any 2×4 matrix of rank 2 can be brought to one of the following matrices

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}, \begin{bmatrix} 1 & * & 0 & * \\ 0 & * & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & 0 \\ 0 & * & * & 1 \end{bmatrix},$$

$$\begin{bmatrix} * & 1 & 0 & * \\ * & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} * & 1 & * & 0 \\ * & 0 & * & 1 \end{bmatrix}, \begin{bmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix},$$

by row operations. Conclude that any point $V \in G(2, 4)$ can be represented by $V = [A]$, where A has the form of one of the matrices above.

(d) Cover $G(2, 4)$ by the sets

$$U_1 = \left\{ V \in G(2, 4) : V = \left[\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix} \right] \right\},$$

$$U_2 = \left\{ V \in G(2, 4) : V = \left[\begin{bmatrix} 1 & * & 0 & * \\ 0 & * & 1 & * \end{bmatrix} \right] \right\},$$

and so on for $U_3, \dots, U_6$. By the argument in (b), these sets are open. Define coordinates $\varphi_1 : U_1 \rightarrow \mathbb{R}^4$ by

$$\varphi_1 \left(\left[\begin{bmatrix} 1 & 0 & x_1 & x_3 \\ 0 & 1 & x_2 & x_4 \end{bmatrix} \right] \right) = (x_1, x_2, x_3, x_4),$$

$$\varphi_2 \left(\left[\begin{bmatrix} 1 & y_1 & 0 & y_3 \\ 0 & y_2 & 1 & y_4 \end{bmatrix} \right] \right) = (y_1, y_2, y_3, y_4),$$

and similarly for the other (U_i, φ_i) . Compute the coordinate change $\varphi_1 \circ \varphi_2^{-1}$.

A similar argument shows that all $\varphi_i \circ \varphi_j^{-1}$ are smooth functions, and hence $G(2, 4)$ is given the structure of a smooth manifold. In fact, this argument can be generalized to the Grassmannian $G(k, n)$, which is the space of k -planes in $\mathbb{R}^n$.

Solution. For part (a), recall that two matrices B and C being row-equivalent means that there exists an invertible matrix P such that

$$B = PC.$$

We write the given 2×4 matrix A in block form:

$$A = \begin{bmatrix} A_{\text{left}} & A_{\text{right}} \end{bmatrix}.$$

If A is row-equivalent to the given matrix, there exists an invertible $P \in \text{Mat}_{2 \times 2}$ such that

$$\begin{bmatrix} A_{\text{left}} & A_{\text{right}} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{21} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 & x_1 & x_3 \\ 0 & 1 & x_2 & x_4 \end{bmatrix}. \quad (4)$$

We see that necessarily $P = A_{\text{left}}$. Therefore P is uniquely determined by the entries of A , and the x_i are found by

$$\begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix} = A_{\text{left}}^{-1} A_{\text{right}}. \quad (5)$$

This shows that the x_i are uniquely determined by the entries of A .

For part (b), suppose once again that (4) holds. Then A_{left} is invertible. Invertibility of a matrix is an open condition: if coefficients of a 2×2 matrix $\tilde{A}_{\text{left}}$ are sufficiently close to A_{left} , then $\tilde{A}_{\text{left}}$ is invertible. Therefore, if the coefficients of a 2×4 matrix

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{\text{left}} & \tilde{A}_{\text{right}} \end{bmatrix}.$$

are sufficiently close to A , then $\tilde{A}_{\text{left}}$ is invertible and we write the analog of (4).

$$\begin{bmatrix} \tilde{A}_{\text{left}} & \tilde{A}_{\text{right}} \end{bmatrix} = \tilde{A}_{\text{left}} \begin{bmatrix} 1 & 0 & \tilde{x}_1 & \tilde{x}_3 \\ 0 & 1 & \tilde{x}_2 & \tilde{x}_4 \end{bmatrix}, \quad \begin{bmatrix} \tilde{x}_1 & \tilde{x}_3 \\ \tilde{x}_2 & \tilde{x}_4 \end{bmatrix} = \tilde{A}_{\text{left}}^{-1} \tilde{A}_{\text{right}}.$$

For part (c), if A has rank 2 then two of its four columns must be linearly independent. The listed matrices are just the ways to select two linearly independent columns from four possibilities. For example, if columns 1 and 3 of a matrix A are linearly independent, then a theorem of linear algebra states that the matrix can be row-reduced to

$$A \rightsquigarrow \begin{bmatrix} 1 & * & 0 & * \\ 0 & * & 1 & * \end{bmatrix}.$$

For part (d), we are asked to compute $\varphi_1 \circ \varphi_2^{-1}$. We have

$$\varphi_2^{-1}(y_1, y_2, y_3, y_4) = \begin{bmatrix} 1 & y_1 & 0 & y_3 \\ 0 & y_2 & 1 & y_4 \end{bmatrix}.$$

We can use formula (5) to find the x_i .

$$\begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & y_1 \\ 0 & y_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & y_3 \\ 1 & y_4 \end{bmatrix}.$$

The result is

$$\begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & -y_1 y_2^{-1} \\ 0 & y_2^{-1} \end{bmatrix} \begin{bmatrix} 0 & y_3 \\ 1 & y_4 \end{bmatrix} = \begin{bmatrix} -y_1 y_2^{-1} & y_3 - y_1 y_4 y_2^{-1} \\ y_2^{-1} & y_4 y_2^{-1} \end{bmatrix},$$

or

$$(x_1, x_2, x_3, x_4) = \left(-\frac{y_1}{y_2}, \frac{1}{y_2}, y_3 - \frac{y_1 y_4}{y_2}, \frac{y_4}{y_2} \right).$$

□

Problem 4. Let

$$S^1 = \{e^{i\theta} : \theta \in [0, 2\pi]\}.$$

Cover S^1 by the following two open sets:

$$U = \{e^{i\theta} : \theta \in (0, 2\pi)\}$$

$$\tilde{U} = \{e^{i\tilde{\theta}} : \tilde{\theta} \in (-\pi, \pi)\}.$$

Use the cover $S^1 = U \cup \tilde{U}$ to equip the tangent bundle TS^1 with coordinates. Compute the change of coordinates on TS^1 , and conclude that TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$.

Solution. The change of coordinates on the circle is

$$\tilde{\theta} = f(\theta) = \begin{cases} \theta, & \text{on } \cap, \\ \theta - 2\pi, & \text{on } \cup. \end{cases}$$

Therefore

$$\frac{\partial \tilde{\theta}}{\partial \theta} = 1.$$

The fiber coordinate q on the tangent bundle is defined by the transformation law

$$\tilde{q} = \frac{\partial \tilde{\theta}}{\partial \theta} q.$$

Therefore $\tilde{q} = q$ is unchanged on the overlap, and the change of coordinates on TS^1 is

$$(\tilde{\theta}, \tilde{q}) = (f(\theta), q).$$

A diffeomorphism

$$f : TS^1 \rightarrow S^1 \times \mathbb{R}$$

can be defined in these coordinates by

$$f(e^{i\theta}, q) = (e^{i\theta}, q),$$

and this formula is well-defined on either coordinate chart. □

Problem 5. Equip S^2 with stereographic coordinates:

$$S^2 = U \cup \tilde{U}, \quad U = S^2 \setminus \{N\}, \quad \tilde{U} = S^2 \setminus \{S\}$$

Let (u^1, u^2) denote stereographic coordinates on U , let $(\tilde{u}^1, \tilde{u}^2)$ denote stereographic coordinates on $\tilde{U}$, and denote the coordinate change by

$$(\tilde{u}^1, \tilde{u}^2) = (f^1(u), f^2(u)).$$

(a) Compute the change of coordinates on the tangent bundle TS^2 . Recall that this is of the form:

$$(\tilde{u}^1, \tilde{u}^2, \tilde{q}^1, \tilde{q}^2) = \left(f^1(u), f^2(u), \frac{\partial \tilde{u}^1}{\partial u^i} q^i, \frac{\partial \tilde{u}^2}{\partial u^i} q^i \right). \quad (6)$$

(b) We can use these calculations to show that every vector field on S^2 must vanish somewhere. Recall that a vector field is a smooth map

$$V : S^2 \rightarrow TS^2$$

satisfying $\pi \circ V = id$, where $\pi : TS^2 \rightarrow S^2$ is the bundle projection locally given by $(u, q) \mapsto u$. In the local coordinates above, a vector field creates a pair of maps

$$\begin{bmatrix} q^1(u) \\ q^2(u) \end{bmatrix} : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad \begin{bmatrix} \tilde{q}^1(\tilde{u}) \\ \tilde{q}^2(\tilde{u}) \end{bmatrix} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad (7)$$

satisfying the change of coordinates relation

$$\tilde{q}^i = \frac{\partial \tilde{u}^i}{\partial u^k} q^k.$$

Show that it is impossible for both functions in (7) to be nowhere zero. The formulas simplify in complex variables. Let

$$z = u^1 + iu^2, \quad \hat{z} = \tilde{u}^1 + i\tilde{u}^2, \quad Q = q^1 + iq^2, \quad \hat{Q} = \tilde{q}^1 + i\tilde{q}^2.$$

The change of coordinates relations (6) on the overlap $\mathbb{C}^*$ are then

$$\hat{z} = \frac{1}{\bar{z}}, \quad \hat{Q} = -\frac{1}{\bar{z}^2} \bar{Q}.$$

Assuming both $Q(z)$ and $\hat{Q}(\hat{z})$ are both nowhere zero smooth functions on $\mathbb{C}$, one approach to a contradiction is to compare the winding number of the image curve $Q(C_r)$, where $C_r = \{|z| = r\}$, for both small and large radius $r > 0$.

Solution. We computed earlier the coordinate change of S^2 . The result is

$$(\tilde{u}^1, \tilde{u}^2) = \left(\frac{u^1}{(u^1)^2 + (u^2)^2}, \frac{u^2}{(u^1)^2 + (u^2)^2} \right). \quad (8)$$

Taking derivatives gives

$$\begin{aligned} \frac{\partial \tilde{u}^1}{\partial u^1} &= \frac{(u^2)^2 - (u^1)^2}{((u^1)^2 + (u^2)^2)^2} \\ \frac{\partial \tilde{u}^1}{\partial u^2} &= -\frac{2u^1 u^2}{((u^1)^2 + (u^2)^2)^2} = \frac{\partial \tilde{u}^2}{\partial u^1} \\ \frac{\partial \tilde{u}^2}{\partial u^2} &= \frac{(u^1)^2 - (u^2)^2}{((u^1)^2 + (u^2)^2)^2}. \end{aligned}$$

Therefore

$$\begin{bmatrix} \tilde{q}^1 \\ \tilde{q}^2 \end{bmatrix} = \frac{1}{((u^1)^2 + (u^2)^2)^2} \begin{bmatrix} (u^2)^2 - (u^1)^2 & -2u^1 u^2 \\ -2u^1 u^2 & (u^1)^2 - (u^2)^2 \end{bmatrix} \begin{bmatrix} q^1 \\ q^2 \end{bmatrix}. \quad (9)$$

These formulas become simpler when written in complex variables. Set

$$z = u^1 + iu^2, \quad \hat{z} = \tilde{u}^1 + i\tilde{u}^2, \quad Q = q^1 + iq^2, \quad \hat{Q} = \tilde{q}^1 + i\tilde{q}^2.$$

Then (8) becomes

$$\hat{z} = \frac{z}{|z|^2},$$

while, (9) is

$$\hat{Q} = -\frac{1}{|z|^4} \bar{z}^2 \bar{Q}$$

using the complex multiplication formula

$$(a + ib)^2(q_1 - iq_2) = \left[(a^2 - b^2)q_1 + 2abq_2 \right] + i \left[2abq_1 + (b^2 - a^2)q_2 \right].$$

The transformation laws (8) and (9) then read

$$\hat{z} = \frac{1}{\bar{z}}, \quad \hat{Q} = -\frac{1}{\bar{z}^2} \bar{Q}. \quad (10)$$

Suppose $Q(z)$ and $\hat{Q}(\hat{z})$ are both nowhere-zero smooth functions on $\mathbb{C}$. We will find a contradiction. Consider $f : (0, \infty) \rightarrow \mathbb{Z}$, defined by

$$f(r) = \frac{1}{2\pi} \Delta_{C_r} Q, \quad C_r = \{|z| = r\}$$

where $\Delta_C Q$ denotes the change in argument of Q around the loop C . In other words, $f(r)$ is the winding number of $Q(C_r)$, so that

$$f(r) = \frac{1}{2\pi i} \int_{Q(C_r)} \frac{d\zeta}{\zeta}.$$

Note that $f(r)$ varies continuously in r , but f is also an integer, hence $f(r)$ must be a constant function. Since $Q(0) \neq 0$, the loop $Q(C_r)$ does not contain the origin and so

$$f(r) \equiv 0$$

for all small r . Therefore $f(r) \equiv 0$ for all $r > 0$. But on the other hand, (10) implies

$$Q(z) = -z^2 \overline{\hat{Q}(1/\bar{z})}, \implies Q(re^{i\theta}) = -r^2 e^{2i\theta} \overline{\hat{Q}(r^{-1}e^{i\theta})}.$$

For sufficiently large $r \gg 1$, then $\hat{Q}(r^{-1}e^{i\theta})$ approaches a non-zero complex number $\hat{Q}(0) \neq 0$ and doesn't wind around the origin, and meanwhile the $e^{2i\theta}$ gives a change in argument of 4π . So $f(r) \equiv 2$ for all large r . This is a contradiction. $\square$

Problem 6. Show that the map $F : S^2 \rightarrow \mathbb{CP}^1$ given by

$$F(x, y, z) = \begin{cases} [x + iy, 1 - z] & \text{if } z \neq 1, \\ [1, 0] & \text{if } (x, y, z) = (0, 0, 1) \end{cases}$$

is a diffeomorphism.

Solution. We open up coordinate charts and take a look. Equip S^2 with coordinates (u, v) over $U = S^2 \setminus \{N\}$, and equip $\mathbb{CP}^1$ with a holomorphic coordinate $z = Z_0/Z_1$ over $U_1 = \{Z_1 \neq 0\}$. By (2), the map

$$F(x, y, z) = F\left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right)$$

evaluates to

$$F(u, v) = \left[\frac{2(u+iv)}{1+u^2+v^2}, \frac{2}{1+u^2+v^2} \right] = [u+iv, 1].$$

Therefore

$$F \stackrel{\text{loc}}{=} u + iv,$$

where the above expression is short-hand notation for $\psi \circ F \circ \varphi^{-1}$ with $\varphi : S^2 \rightarrow \mathbb{R}^2$ a stereographic chart and $\psi : \mathbb{CP}^1 \rightarrow \mathbb{C}$ a holomorphic chart. After identifying $\mathbb{C} \cong \mathbb{R}^2$, we see that F is, over this coordinate patch, the identity $(u, v) \mapsto (u, v)$. Thus

$$F : S^2 \setminus \{N\} \rightarrow \mathbb{CP}^1 \setminus \{[1, 0]\}$$

is a smooth bijection. Putting back the missing point $F(N) = [1, 0]$ makes

$$F : S^2 \rightarrow \mathbb{CP}^1$$

a bijection. It remains to show that F and its inverse are smooth near N . For this, we use coordinates $(\tilde{u}, \tilde{v})$ over $\tilde{U} = S^2 \setminus \{S\}$ and coordinates $\tilde{z} = Z_1/Z_0$ over $U_0 = \{Z_0 \neq 0\}$. By (3), the map

$$F(x, y, z) = F\left(\frac{2\tilde{u}}{1 + \tilde{u}^2 + \tilde{v}^2}, \frac{2\tilde{v}}{1 + \tilde{u}^2 + \tilde{v}^2}, \frac{1 - \tilde{u}^2 - \tilde{v}^2}{1 + \tilde{u}^2 + \tilde{v}^2}\right)$$

evaluates to

$$F(\tilde{u}, \tilde{v}) = \left[\frac{2(\tilde{u} + i\tilde{v})}{1 + \tilde{u}^2 + \tilde{v}^2}, \frac{2(\tilde{u}^2 + \tilde{v}^2)}{1 + \tilde{u}^2 + \tilde{v}^2} \right] = [\tilde{u} + i\tilde{v}, \tilde{u}^2 + \tilde{v}^2].$$

For a non-zero complex number $a + ib$, the ratio $|a + ib|^2/(a + ib)$ is just $a - ib$. Therefore

$$F \stackrel{\text{loc}}{=} \tilde{u} - i\tilde{v}.$$

This local formula extends across zero, and agrees with $F(N) = [1, 0]$. We see that F and its inverse are smooth near N . In sum, we have shown that $F : S^2 \rightarrow \mathbb{CP}^1$ is a bijection and also a local diffeomorphism everywhere. Therefore F is a diffeomorphism. $\square$

Problem 7. Consider the map $F : S^2 \rightarrow \mathbb{RP}^2$ given by

$$F(x, y, z) = [x, y, z].$$

Show that F is smooth. Is F surjective? Is F injective?

Solution. The map is not injective since

$$F(x, y, z) = F(-x, -y, -z).$$

More precisely, if $F(p) = F(q)$, then $p = \lambda q$ for some $\lambda \in \mathbb{R}$. Since $|p| = |\lambda||q|$ and $p, q \in \mathbb{S}^2$ then $|\lambda| = 1$ and $\lambda = \pm 1$. Thus F is 2 : 1.

The map F is surjective, since given $[x, y, z] \in \mathbb{RP}^2$, we can attain it by

$$F\left(\frac{1}{\sqrt{x^2 + y^2 + z^2}}(x, y, z)\right) = [x, y, z].$$

For smoothness, we open up coordinate charts and take a look. Equip S^2 with coordinates (u, v) over $U = S^2 \setminus \{N\}$. By (2), the map

$$F(x, y, z) = F\left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right)$$

becomes

$$F(u, v) = [2u, 2v, u^2 + v^2 - 1].$$

We can now choose coordinates on $\mathbb{RP}^2$. If $u \neq 0$, then we can use the chart $U_0 = \{X_0 \neq 0\}$ with coordinates $(w^1, w^2) = (X_1/X_0, X_2/X_0)$. The local form of the map is then

$$F \stackrel{\text{loc}}{=} \left(\frac{v}{u}, \frac{u^2 + v^2 - 1}{2u}\right),$$

which is smooth. If $v \neq 0$, then the chart $U_1 = \{X_1 \neq 0\}$ can be used. If $u = v = 0$, the chart $U_2 = \{X_2 \neq 0\}$ can be used. Finally, one should also check smoothness near N using the coordinates $(\tilde{u}, \tilde{v})$ over $\tilde{U} = S^2 \setminus \{S\}$. We omit the local expressions for F for all these combinations since it is tedious. $\square$

Problem 8. Show that $F : S^1 \rightarrow \mathbb{RP}^1$ given by

$$F(e^{i\theta}) = [\cos(\theta/2), \sin(\theta/2)]$$

is a diffeomorphism.

Solution. First, we must show that F is well-defined. Since the same point in S^1 can appear as $e^{i\theta}$ or $e^{i(\theta+2\pi k)}$ for an integer k , we check

$$F(e^{i(\theta+2\pi k)}) = [\cos(\theta/2 + k\pi), \sin(\theta/2 + k\pi)] = [-\cos(\theta/2), -\sin(\theta/2)] = F(e^{i\theta}).$$

Therefore F is well-defined. To construct an inverse, we note that

$$\cos(\theta/2) + i\sin(\theta/2) = e^{i\theta/2}$$

squares back to $e^{i\theta}$.

$$\left[\cos(\theta/2) + i\sin(\theta/2)\right]^2 = e^{i\theta}. \quad (11)$$

Therefore to undo the $\theta/2$, we convert $[x, y]$ to a unit vector and define

$$F^{-1}[x, y] = \left[\frac{x}{\sqrt{x^2 + y^2}} + i\frac{y}{\sqrt{x^2 + y^2}}\right]^2.$$

The map

$$F^{-1} : \mathbb{RP}^1 \rightarrow S^1$$

is well-defined, since if $[x, y] = [\lambda x, \lambda y]$, then

$$F^{-1}[\lambda x, \lambda y] = \frac{(x + iy)^2}{x^2 + y^2} = F^{-1}[x, y].$$

It is quick to check using (11) that

$$F^{-1} \circ F(e^{i\theta}) = e^{i\theta}, \quad F \circ F^{-1}[x, y] = [x, y].$$

Therefore F is bijective and it remains to show that F and F^{-1} are smooth.

- In coordinates $U = \{e^{i\theta} : \theta \in (0, 2\pi)\} \subseteq S^1$, the image of F lands in $V = \{[x, y] : y \neq 0\} \subseteq \mathbb{RP}^2$, and using these coordinates

$$F \stackrel{\text{loc}}{=} \frac{\cos(\theta/2)}{\sin(\theta/2)}.$$

Since

$$F'(\theta) = -\frac{1}{2 \sin^2(\theta/2)},$$

we have that $F' \neq 0$ over $(0, 2\pi)$. By the inverse function theorem, F is a local diffeomorphism over this chart.

- In coordinates $\tilde{U} = \{e^{i\tilde{\theta}} : \tilde{\theta} \in (-\pi, \pi)\} \subseteq S^1$, the image of F lands in $\tilde{V} = \{[x, y] : x \neq 0\} \subseteq \mathbb{RP}^2$, and using these coordinates

$$F \stackrel{\text{loc}}{=} \frac{\sin(\tilde{\theta}/2)}{\cos(\tilde{\theta}/2)}.$$

Since

$$F'(\tilde{\theta}) = \frac{1}{2 \cos^2(\tilde{\theta}/2)},$$

we have that $F' \neq 0$ over $(-\pi, \pi)$. By the inverse function theorem, F is a local diffeomorphism over this chart.

In sum, F is a bijection and a local diffeomorphism. Therefore F is a diffeomorphism. $\square$