

Problem Set 2: Submanifolds, Submersions, and Immersions

Problem 1. Let

$$\pi : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{RP}^2, \quad (x, y, z) \mapsto [x, y, z]$$

be the quotient map. Show that π is a submersion.

Problem 2. Let $\gamma : (-\pi, \pi) \rightarrow \mathbb{S}^2$ be given by

$$\gamma(t) = \left(\frac{1}{\sqrt{2}} \sin(2t), \frac{1}{\sqrt{2}} \sin t, \sqrt{1 - \frac{1}{2} \sin^2(2t) - \frac{1}{2} \sin^2 t} \right).$$

Show that γ is an immersion.

Problem 3. Let $f : \mathbb{RP}^1 \rightarrow \mathbb{RP}^2$ be given by

$$f[x, y] = [x^2, xy, y^2].$$

Show that f is an immersion.

Problem 4. Let $F : S^2 \rightarrow \mathbb{R}^2$ be the map $(x, y, z) \mapsto (x, y)$. Let $p = (x_0, y_0, z_0) \in S^2$. Compute $dF|_p$ acting on

$$v \in T_p S^2 = \{v \in \mathbb{R}^3 : v \cdot p = 0\}$$

and find all points $p \in S^2$ such that $dF|_p$ has a nontrivial kernel.

Problem 5. Let M be a smooth manifold and $p \in M$. Show that

$$S = \{p\} \subset M$$

is a submanifold.

Problem 6. Let $f : M \rightarrow N$ be a smooth map. Let $p \in M$ and $c = f(p)$. Suppose $df|_p : T_p M \rightarrow T_{f(p)} N$ is surjective. Is $f^{-1}(c) \subseteq M$ a submanifold?

Problem 7. Suppose $\tilde{F} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a smooth map. Suppose

$$M \subseteq \mathbb{R}^m, \quad N \subseteq \mathbb{R}^n$$

are submanifolds with $\tilde{F}(M) \subseteq N$. Show that

$$F : M \rightarrow N$$

defined by $F = \tilde{F}|_M$ is a smooth map.

Problem 8. Let $P : \mathbb{R}^k \rightarrow \mathbb{R}$ be a homogeneous polynomial of degree m . This means

$$P(tx_1, \dots, tx_k) = t^m P(x_1, \dots, x_k).$$

Assume m is an integer with $m \geq 2$.

(a) Prove Euler's identity

$$\sum_{i=1}^k x_i \frac{\partial P}{\partial x_i} = mP.$$

Hint: consider $\frac{d}{dt} \Big|_{t=1} P(tx) = \frac{d}{dt} \Big|_{t=1} t^m P(x)$.

(b) Let $a > 0$. Prove that

$$X_a = \{x \in \mathbb{R}^k : P(x) = a\}$$

is a $k - 1$ dimensional submanifold of $\mathbb{R}^k$.

(c) Prove that X_a is diffeomorphic to X_1 .

Problem 9. Consider

$$S = \{(e^{i\theta}, e^{2i\theta}) : \theta \in [0, 2\pi]\} \subset T^2$$

where $T^2 = S^1 \times S^1$. Show that $S \subset T^2$ is a submanifold.

Problem 10. Let $F = (f, g) : \mathbb{R}^n \rightarrow \mathbb{R}^2$ be a smooth function. Let α be a regular value of f and let (α, β) be a regular value of F .

(a) Let $Y = f^{-1}(\alpha)$ and $Z = F^{-1}(\alpha, \beta)$. Prove that $Z \subset Y$ is a submanifold of Y .

(b) Consider the function $h : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ given by

$$h(x, y, z, w) = (x^2 + y^2, z^2 + w^2).$$

For any $\alpha \in (0, 1)$, show that $h^{-1}(\alpha, 1 - \alpha)$ is a 2-dimensional submanifold of S^3 .

Problem 11. Let $p : X \rightarrow B$ be a surjective submersion such that $p^{-1}(b)$ is connected for all $b \in B$. Show that if $f \in C^\infty(X)$ satisfies

$$\xi(f) = 0, \quad \text{for all } x \in X \text{ and } \xi \in \ker(dp|_x),$$

then f descends to a function on the base B . In more detail, this means there exists $\bar{f} \in C^\infty(B)$ such that $f = \bar{f} \circ p$.

Problem 12. Let $p : X \rightarrow B$ be a submersion and $q : Y \rightarrow B$ a smooth map. The fiber product is defined by

$$X \times_B Y = \left\{ (x, y) \in X \times Y : p(x) = q(y) \right\}.$$

(a) Let $A : V \rightarrow W$ be a surjective linear map of vector spaces $V \cong \mathbb{R}^n$, $W \cong \mathbb{R}^m$. What is $V \times_W 0$?

(b) Let $p : \mathbb{R}^2 \rightarrow \mathbb{R}$ be $p(x, v) = x$. What is $\mathbb{R}^2 \times_{\mathbb{R}} \mathbb{R}^2$?

(c) Show that $X \times_B Y$ is a manifold by realizing it as a submanifold of $X \times Y$.

(d) Without the submersion hypothesis, the set $X \times_B Y$ is not necessarily a manifold. If $p : \mathbb{R} \rightarrow \mathbb{R}$ is $p(x) = x^2$, what is $\mathbb{R} \times_{\mathbb{R}} \mathbb{R}$?

Problem 13. Let

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$$

and define

$$p : S^3 \rightarrow \mathbb{CP}^1, \quad p(z_1, z_2) = [z_1, z_2].$$

This is the Hopf fibration.

(a) Show that p is a submersion. Here are some useful coordinates. Compute near a given $(z_1, z_2) \in S^3$. One of z_1 or z_2 is non-zero, and let us simply assume that $z_1 \neq 0$. We can then write $z_1 = |z_1|e^{i\theta}$. Over $\{z_1 \neq 0\}$, use the complex coordinate

$$u = \frac{z_2}{z_1}.$$

You can then write a point $(z_1, z_2) \in \{z_1 \neq 0\} \cap S^3$ as

$$(z_1, z_2) = \frac{e^{i\theta}}{\sqrt{1 + |u|^2}}(1, u).$$

Write the map p in the local coordinates $(\theta, \operatorname{Re} u, \operatorname{Im} u)$ to see its submersion structure.

(b) Consider the four-manifold

$$M = S^3 \times_{\mathbb{CP}^1} S^3.$$

Let $\pi : M \rightarrow \mathbb{CP}^1$ be its projection to the base $(x, y) \mapsto p(x)$. Identify the submanifolds $\pi^{-1}(b)$ for $b \in \mathbb{CP}^1$.
