

Problem Set 2: Submanifolds, Submersions, and Immersions

Problem 1. Let

$$\pi : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{RP}^2, \quad (x, y, z) \mapsto [x, y, z]$$

be the quotient map. Show that π is a submersion.

Solution. Let $(x_0, y_0, z_0) \in \mathbb{R}^3 \setminus \{0\}$. Up to relabelling, let us assume that $z_0 \neq 0$. Let (u, v) be coordinates over $U \subseteq \mathbb{RP}^2$, where $U = \{[x, y, z] : z \neq 0\}$, given by $u = x/z$ and $v = y/z$. The coordinate representation of π is then

$$\pi(x, y, z) \stackrel{\text{loc}}{=} \left(\frac{x}{z}, \frac{y}{z} \right).$$

Its derivative matrix is

$$d\pi|_{(x_0, y_0, z_0)} = \begin{bmatrix} \frac{1}{z_0} & 0 & -\frac{x_0}{z_0^2} \\ 0 & \frac{1}{z_0} & -\frac{y_0}{z_0^2} \end{bmatrix}.$$

The first two columns are linearly independent hence $d\pi|_{(x_0, y_0, z_0)}$ is surjective. Therefore π is a submersion. $\square$

Problem 2. Let $\gamma : (-\pi, \pi) \rightarrow \mathbb{S}^2$ be given by

$$\gamma(t) = \left(\frac{1}{\sqrt{2}} \sin(2t), \frac{1}{\sqrt{2}} \sin t, \sqrt{1 - \frac{1}{2} \sin^2(2t) - \frac{1}{2} \sin^2 t} \right).$$

Show that γ is an immersion.

Solution. The curve γ lies on the upper hemisphere. Let (x, y) be local coordinates over $\{z > 0\} \subseteq \mathbb{S}^2$. The coordinate representation of γ is then

$$\gamma(t) \stackrel{\text{loc}}{=} \frac{1}{\sqrt{2}} (\sin 2t, \sin t).$$

Let $t_0 \in (-\pi, \pi)$. The derivative matrix in these coordinates is

$$d\gamma_{t_0}(a) = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 \cos 2t_0 \\ \cos t_0 \end{bmatrix} a.$$

We note that

$$\begin{bmatrix} 2 \cos 2t_0 \\ \cos t_0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

since if $\cos t_0 = 0$ then $t_0 = \pm\pi/2$ and $2 \cos 2t_0 = -2$. Therefore

$$\ker d\gamma_{t_0} = \{0\}$$

and hence the linear map $d\gamma_{t_0} : \mathbb{R} \rightarrow \mathbb{R}^2$ is injective, and so γ is an immersion. $\square$

Problem 3. Let $f : \mathbb{RP}^1 \rightarrow \mathbb{RP}^2$ be given by

$$f[x, y] = [x^2, xy, y^2].$$

Show that f is an immersion.

Solution. First, we note that f is well-defined since if (x, y) is replaced by $(\lambda x, \lambda y)$, then

$$f[\lambda x, \lambda y] = f[x, y].$$

To show f is an immersion, we compute locally near a given point $[x_0, y_0] \in \mathbb{RP}^1$. Either $x_0 \neq 0$ or $y_0 \neq 0$, and let us assume for concreteness that $x_0 \neq 0$. Let $x_0 \in U \subseteq \mathbb{RP}^1$ be the open neighborhood

$$U = \{[x, y] \in \mathbb{RP}^1 : x \neq 0\}$$

with local coordinates $u = y/x$ over U . On the target, let $f(x_0) \in V \subseteq \mathbb{RP}^2$ be the open neighborhood

$$V = \{[x, y, z] \in \mathbb{RP}^2 : x \neq 0\}$$

with local coordinates

$$v^1 = \frac{y}{x}, \quad v^2 = \frac{z}{x}.$$

In coordinates (U, u) and (V, v^i) , the map f has the local smooth representation

$$f(u) \stackrel{\text{loc}}{=} (u, u^2).$$

The derivative matrix over U is

$$df|_u = \begin{bmatrix} 1 \\ 2u \end{bmatrix}.$$

This matrix is injective at any point u , hence f is an immersion. $\square$

Problem 4. Let $F : S^2 \rightarrow \mathbb{R}^2$ be the map $(x, y, z) \mapsto (x, y)$. Let $p = (x_0, y_0, z_0) \in S^2$. Compute $dF|_p$ acting on

$$v \in T_p S^2 = \{v \in \mathbb{R}^3 : v \cdot p = 0\}$$

and find all points $p \in S^2$ such that $dF|_p$ has a nontrivial kernel.

Solution. Let $\gamma(t)$ be a path on S^2 with $\gamma(0) = (x_0, y_0, z_0) = p$ and $\dot{\gamma}(0) = (v_1, v_2, v_3)$. Then

$$dF|_p v = \left. \frac{d}{dt} \right|_{t=0} F \circ \gamma = (v_1, v_2).$$

If $v \in \ker dF|_p$, then $v_1 = v_2 = 0$, and the condition that $v \in T_p S^2$ becomes

$$v \cdot (x_0, y_0, z_0) = 0 \implies v_3 z_0 = 0.$$

If $z_0 \neq 0$, then $v_3 = 0$ and so $\ker dF|_p = \{0\}$. If $z_0 = 0$, then

$$\ker dF|_p = \{(0, 0, v_3) : v_3 \in \mathbb{R}\}.$$

Therefore $dF|_p$ only has a nontrivial kernel on the equator $\{z = 0\} \subseteq S^2$. □

Problem 5. Let M be a smooth manifold and $p \in M$. Show that

$$S = \{p\} \subset M$$

is a submanifold.

Solution. Let $(U, \varphi) \subset M$ be a coordinate chart with $p \in U$ and $\varphi(p) = 0$. Then $\varphi(U \cap \{p\}) = \{(x^1, \dots, x^n) = 0\}$. This satisfies the definition of a submanifold. □

Problem 6. Let $f : M \rightarrow N$ be a smooth map. Let $p \in M$ and $c = f(p)$. Suppose $df|_p : T_p M \rightarrow T_{f(p)} N$ is surjective. Is $f^{-1}(c) \subseteq M$ a submanifold?

Solution. No. To apply the submersion theorem and conclude that $f^{-1}(c)$ is a submanifold, we would need that $df|_p : T_p M \rightarrow T_{f(p)} N$ is surjective for all $p \in f^{-1}(c)$, not just a single one. A counterexample is

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad (x, y) \mapsto xy.$$

Let $p = (1, 0)$ and $f(p) = 0$. Then

$$df|_p = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

is surjective, but $f^{-1}(0) = \{x = 0\} \cup \{y = 0\}$ is the union of the two coordinate axes and does not have the structure of a submanifold near the origin. $\square$

Problem 7. Suppose $\tilde{F} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a smooth map. Suppose

$$M \subseteq \mathbb{R}^m, \quad N \subseteq \mathbb{R}^n$$

are submanifolds with $\tilde{F}(M) \subseteq N$. Show that

$$F : M \rightarrow N$$

defined by $F = \tilde{F}|_M$ is a smooth map.

Solution. To check that these maps are smooth, we open up a local coordinate chart around $p \in M$. The submanifold property gives the existence of an open neighborhood $p \in U \subseteq \mathbb{R}^m$ and local coordinates $(x, y) \in \mathbb{R}^{\dim M} \times \mathbb{R}^{m - \dim M}$ such that

$$U \cap M = \{(x, y) : y = 0\}.$$

Similarly, around $F(p) \in N$, there is a neighborhood $V \subseteq \mathbb{R}^n$ with coordinates such that

$$V \cap N = \{(u, v) : v = 0\}$$

where $(u, v) \in \mathbb{R}^{\dim N} \times \mathbb{R}^{n - \dim N}$. In these coordinates, the property $\tilde{F}(M) \subseteq N$ gives the local form

$$\tilde{F}(x, 0) \stackrel{\text{loc}}{=} \left(\tilde{F}^1(x, 0), \dots, \tilde{F}^{\dim N}(x, 0), 0, \dots, 0 \right).$$

Therefore the local coordinate representation of $F : M \rightarrow N$ near p is

$$F(x) \stackrel{\text{loc}}{=} \left(\tilde{F}^1(x, 0), \dots, \tilde{F}^{\dim N}(x, 0) \right).$$

This local expression for F is smooth. Since $p \in M$ is arbitrary, then F is smooth. $\square$

Problem 8. Let $P : \mathbb{R}^k \rightarrow \mathbb{R}$ be a homogeneous polynomial of degree m . This means

$$P(tx_1, \dots, tx_k) = t^m P(x_1, \dots, x_k).$$

Assume m is an integer with $m \geq 2$.

(a) Prove Euler's identity

$$\sum_{i=1}^k x_i \frac{\partial P}{\partial x_i} = mP.$$

Hint: consider $\frac{d}{dt}\big|_{t=1} P(tx) = \frac{d}{dt}\big|_{t=1} t^m P(x)$.

(b) Let $a > 0$. Prove that

$$X_a = \{x \in \mathbb{R}^k : P(x) = a\}$$

is a $k - 1$ dimensional submanifold of $\mathbb{R}^k$.

(c) Prove that X_a is diffeomorphic to X_1 .

Solution. We start with part (a). By the chain rule, we have

$$\frac{d}{dt}\bigg|_{t=1} P(tx) = \sum_{i=1}^k x_i \frac{\partial P}{\partial x_i}.$$

On the other hand, we have

$$\frac{d}{dt}\bigg|_{t=1} t^m P(x) = mP(x).$$

Equating these two expressions proves Euler's identity. Next, we consider part (b). Let $x \in X_a$. Euler's identity implies that if

$$dP|_x = \left(\frac{\partial P}{\partial x_1}, \dots, \frac{\partial P}{\partial x_k} \right) = (0, \dots, 0)$$

then $P(x) = 0$. Since $P(x) = a \neq 0$, we conclude that one of the partials is non-zero, and hence $dP|_x$ is surjective. Therefore a is a regular value, and by the regular value level set theorem we see that X_a is a submanifold of dimension $k - 1$. Finally, we discuss part (c). Consider the map $F : X_a \rightarrow X_1$ given by

$$F(x_1, \dots, x_k) = a^{-1/m}(x_1, \dots, x_k).$$

Its inverse is evidently

$$F^{-1}(x_1, \dots, x_k) = a^{1/m}(x_1, \dots, x_k).$$

These maps are smooth by the following fact: if $\tilde{F} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is smooth and $M \subseteq \mathbb{R}^m$, $N \subseteq \mathbb{R}^n$ are submanifolds with $\tilde{F}(M) \subseteq N$, then $F : M \rightarrow N$ defined by $F = \tilde{F}|_M$ is again smooth. $\square$

Problem 9. Consider

$$S = \{(e^{i\theta}, e^{2i\theta}) : \theta \in [0, 2\pi]\} \subset T^2$$

where $T^2 = S^1 \times S^1$. Show that $S \subset T^2$ is a submanifold.

Solution. Denote the angle on the first S^1 by $e^{i\theta}$ and the angle on the second S^1 by $e^{i\varphi}$. We cover T^2 by coordinates

$$T^2 = (U_1 \times U_2) \cup (\tilde{U}_1 \times U_2) \cup (U_1 \times \tilde{U}_2) \cup (\tilde{U}_1 \times \tilde{U}_2)$$

where

$$\begin{aligned} U_1 &= \{e^{i\theta} : \theta \in (0, 2\pi)\}, & \tilde{U}_1 &= \{e^{i\tilde{\theta}} : \tilde{\theta} \in (-\pi, \pi)\} \\ U_2 &= \{e^{i\varphi} : \varphi \in (0, 2\pi)\}, & \tilde{U}_2 &= \{e^{i\tilde{\varphi}} : \tilde{\varphi} \in (-\pi, \pi)\}. \end{aligned}$$

Let $p = (e^{i\theta}, e^{2i\theta}) \in S$. Let us just assume $\theta \in (0, \pi)$ to be concrete; the other cases are similar but with different labeling. Consider the following open neighborhood $W \subseteq T^2$ of p .

$$W = \{(e^{i\theta}, e^{i\varphi}) : \theta \in (0, \pi), \varphi \in (0, 2\pi)\}.$$

Then

$$S \cap W = \{(\theta, \varphi) \in W : \varphi = 2\theta\}.$$

After the change of variables

$$x = \theta, \quad y = \varphi - 2\theta$$

we obtain

$$S \cap W = \{(x, y) : y = 0\}.$$

This proves the embedding property near p .

□

Problem 10. Let $F = (f, g) : \mathbb{R}^n \rightarrow \mathbb{R}^2$ be a smooth function. Let α be a regular value of f and let (α, β) be a regular value of F .

(a) Let $Y = f^{-1}(\alpha)$ and $Z = F^{-1}(\alpha, \beta)$. Prove that $Z \subset Y$ is a submanifold of Y .

(b) Consider the function $h : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ given by

$$h(x, y, z, w) = (x^2 + y^2, z^2 + w^2).$$

For any $\alpha \in (0, 1)$, show that $h^{-1}(\alpha, 1 - \alpha)$ is a 2-dimensional submanifold of S^3 .

Solution. Consider the map $G : Y \rightarrow \mathbb{R}$ given by $G = g|_Y$. We will show that β is a regular value of G .

Let $p \in Y$ with $G(p) = \beta$. Since α is a regular value of f , the submersion theorem gives the existence of a p -neighborhood $U \subseteq \mathbb{R}^n$ with local coordinates $(x_1, \dots, x_n)$ such that

$$f \stackrel{\text{loc}}{=} x_1 + \alpha.$$

It follows that

$$Y \cap U = \{(0, x_2, \dots, x_n)\}.$$

The coordinate representation of G is then

$$G(x_2, \dots, x_n) \stackrel{\text{loc}}{=} g(0, x_2, \dots, x_n).$$

In these coordinates on $\mathbb{R}^n$ it takes the form $F = (x_1 + \alpha, g(x))$ and its derivative is

$$dF|_p = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \partial_1 g & \partial_2 g & \cdots & \partial_n g \end{bmatrix}.$$

Next, we know that F has maximal rank. Thus there must be one $\partial_i g \neq 0$ with $i \geq 2$. Therefore

$$dG|_p = (\partial_2 g, \partial_3 g, \dots, \partial_n g)$$

is surjective. It follows that $G : Y \rightarrow \mathbb{R}$ has β as a regular value. Therefore $Z = G^{-1}(\beta)$ is a submanifold of Y . Note that $Z = F^{-1}(\alpha, \beta)$.

For part (b), let

$$F = (f, g), \quad f = x^2 + y^2 + z^2 + w^2, \quad g = x^2 + y^2$$

If $f(p) = 1$, $g(p) = \alpha$, then $z^2 + w^2 = 1 - \alpha$. Since 1 is a regular value of f and $(1, \alpha)$ a regular value of F , we get the desired result by part (a). □

Problem 11. Let $p : X \rightarrow B$ be a surjective submersion such that $p^{-1}(b)$ is connected for all $b \in B$. Show that if $f \in C^\infty(X)$ satisfies

$$\xi(f) = 0, \quad \text{for all } x \in X \text{ and } \xi \in \ker(dp|_x),$$

then f descends to a function on the base B . In more detail, this means there exists $\bar{f} \in C^\infty(B)$ such that $f = \bar{f} \circ p$.

Solution. We explore the behavior of f locally near a point $x_0 \in X$. Apply the submersion theorem to find a neighborhood $U \subseteq X$ and local coordinates (b, v) on X near $x_0 = (b_0, v_0)$ such that

$$p(b, v) \stackrel{\text{loc}}{=} b.$$

Here $b \in \mathbb{R}^{\dim B}$ and $v \in \mathbb{R}^k$ with $\dim X = \dim B + k$. The derivative of p in these coordinates is

$$dp|_{(b,v)} = \begin{bmatrix} I & 0 \end{bmatrix}.$$

Therefore the kernel is generated by

$$\ker dp|_{(b,v)} = \text{span} \left\{ \frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^k} \right\}.$$

The condition $\xi(f) = 0$, in this local neighborhood U , becomes

$$\frac{\partial f}{\partial v^i} = 0, \quad i \in \{1, \dots, k\}.$$

Hence

$$f(b, v) \in C^\infty(U)$$

depends only on b .

Next, fix $b \in B$. Cover $p^{-1}(b)$ with such neighborhoods U as above, and conclude that

$$f|_{p^{-1}(b)}$$

is a locally constant function. Since $p^{-1}(b)$ is connected, we conclude that $f|_{p^{-1}(b)}$ is constant. This is the statement

$$f(x_b) = f(\tilde{x}_b) \quad \text{for all } x_b, \tilde{x}_b \in p^{-1}(b). \quad (1)$$

We can then define

$$\bar{f} : B \rightarrow \mathbb{R}, \quad \bar{f}(b) = f(x_b)$$

where $x_b \in X$ is any point, which exists by surjectivity, such that $p(x_b) = b$. This function $\bar{f}$ is well-defined by (1). The function $\bar{f}$ is smooth, since as discussed above it locally takes the form

$$\bar{f}(b) \stackrel{\text{loc}}{=} f(b, v_0)$$

in a neighborhood of $b_0 \in B$. Finally, by construction $f = \bar{f} \circ p$.

□

Problem 12. Let $p : X \rightarrow B$ be a submersion and $q : Y \rightarrow B$ a smooth map. The fiber product is defined by

$$X \times_B Y = \left\{ (x, y) \in X \times Y : p(x) = q(y) \right\}.$$

(a) Let $A : V \rightarrow W$ be a surjective linear map of vector spaces $V \cong \mathbb{R}^n$, $W \cong \mathbb{R}^m$. What is $V \times_W 0$?

(b) Let $p : \mathbb{R}^2 \rightarrow \mathbb{R}$ be $p(x, v) = x$. What is $\mathbb{R}^2 \times_{\mathbb{R}} \mathbb{R}^2$?

(c) Show that $X \times_B Y$ is a manifold by realizing it as a submanifold of $X \times Y$.

(d) Without the submersion hypothesis, the set $X \times_B Y$ is not necessarily a manifold. If $p : \mathbb{R} \rightarrow \mathbb{R}$ is $p(x) = x^2$, what is $\mathbb{R} \times_{\mathbb{R}} \mathbb{R}$?

Solution. For part (a), the definition gives

$$V \times_W 0 = \ker A.$$

For part (b), the definition gives

$$\mathbb{R}^2 \times_{\mathbb{R}} \mathbb{R}^2 = \left\{ ((x, v_1), (x, v_2)) \right\} \cong \mathbb{R}^3,$$

so that there are two independent fiber directions v and one base direction x .

For part (c), let $(x_0, y_0) \in X \times_B Y$. Choose coordinates $U \subseteq B$ near $b_0 = p(x_0) = q(y_0)$, so that we may identify $p(x), q(y) \in \mathbb{R}^b$, where b is the dimension of B , for all x close to x_0 and y close to y_0 . We will work in the (x_0, y_0) -neighborhood $\mathcal{U} \subseteq X \times Y$ given by

$$\mathcal{U} = p^{-1}(U) \times q^{-1}(U)$$

where we can define

$$F : \mathcal{U} \rightarrow \mathbb{R}^b$$

by

$$F(x, y) = p(x) - q(y).$$

Then

$$(X \times_B Y) \cap \mathcal{U} = F^{-1}(0),$$

so to show that $X \times_B Y$ is a submanifold near (x_0, y_0) we must show that $dF|_{(x_0, y_0)}$ is surjective.

$$dF|_{(x_0, y_0)} = \begin{bmatrix} dp|_{x_0} & -dq_{y_0} \end{bmatrix}.$$

Since p is a submersion, then $dp|_{x_0}$ is surjective, hence $dF|_{(x_0, y_0)}$ is surjective. Therefore

$$F^{-1}(0) \subseteq X \times Y$$

is a submanifold, and hence $X \times_B Y$ has the local structure of a submanifold near (x_0, y_0) . The point was arbitrary, hence $X \times_B Y$ is a submanifold.

For part (d), the definition gives

$$\mathbb{R} \times_{\mathbb{R}} \mathbb{R} = \{(x, y) : x^2 = y^2\}.$$

Therefore

$$\mathbb{R} \times_{\mathbb{R}} \mathbb{R} = \{y = x\} \cup \{y = -x\},$$

which does not have the structure of a manifold at the origin. □

Problem 13. Let

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$$

and define

$$p : S^3 \rightarrow \mathbb{CP}^1, \quad p(z_1, z_2) = [z_1, z_2].$$

This is the Hopf fibration.

(a) Show that p is a submersion. Here are some useful coordinates. Compute near a given $(z_1, z_2) \in S^3$. One of z_1 or z_2 is non-zero, and let us simply assume that $z_1 \neq 0$. We can then write $z_1 = |z_1|e^{i\theta}$. Over $\{z_1 \neq 0\}$, use the complex coordinate

$$u = \frac{z_2}{z_1}.$$

You can then write a point $(z_1, z_2) \in \{z_1 \neq 0\} \cap S^3$ as

$$(z_1, z_2) = \frac{e^{i\theta}}{\sqrt{1 + |u|^2}}(1, u).$$

Write the map p in the local coordinates $(\theta, \operatorname{Re} u, \operatorname{Im} u)$ to see its submersion structure.

(b) Consider the four-manifold

$$M = S^3 \times_{\mathbb{CP}^1} S^3.$$

Let $\pi : M \rightarrow \mathbb{CP}^1$ be its projection to the base $(x, y) \mapsto p(x)$. Identify the submanifolds $\pi^{-1}(b)$ for $b \in \mathbb{CP}^1$.

Solution. Over $\{z_1 \neq 0\} \cap S^3$, we write $z_1 = |z_1|e^{i\theta}$ and $u = z_2/z_1$ so that

$$(z_1, z_2) = |z_1|e^{i\theta}(1, u). \quad (2)$$

Since $|z_1|^2 + |z_2|^2 = 1$ implies

$$1 = |z_1|^2(1 + |u|^2) \implies |z_1| = \frac{1}{\sqrt{1 + |u|^2}}.$$

we can write (2) as

$$(z_1, z_2) = \frac{e^{i\theta}}{\sqrt{1 + |u|^2}}(1, u).$$

This produces a local map

$$\phi : \{z_1 \neq 0\} \rightarrow S^1 \times \mathbb{C}, \quad (z_1, z_2) \mapsto (e^{i\theta}, u).$$

Shrinking $U \subseteq \{z_1 \neq 0\}$ to be within a chart on S^1 , we obtain a local coordinate chart

$$\varphi : U \rightarrow \mathbb{R}^3, \quad (z_1, z_2) \mapsto (\theta, \operatorname{Re} u, \operatorname{Im} u).$$

Therefore

$$p \circ \varphi^{-1}(\theta, u) = \frac{e^{i\theta}}{\sqrt{1 + |u|^2}}[1, u] = [1, u].$$

Next, we write the map $p : S^3 \rightarrow \mathbb{CP}^1$ using these coordinates on S^3 , and the usual coordinates $u = z_2/z_1$ on the target $\mathbb{CP}^1$. The local representation of p is

$$p(\theta, u) \stackrel{\text{loc}}{=} u.$$

This is the local form of a submersion.

For part (b), we let

$$M = S^3 \times_{\mathbb{CP}^1} S^3, \quad \pi : M \rightarrow \mathbb{CP}^1$$

and explore π in local coordinates. Let $x_0 \in M$ with $\pi(x_0) = b_0$, and use the product of the local coordinates $(e^{i\theta}, u)$ on S^3 , in which $p(e^{i\theta}, u) = u$. Then

$$x_0 = \left((e^{i\theta_0}, b_0), (e^{i\hat{\theta}_0}, b_0) \right)$$

and

$$\pi \left((e^{i\theta}, b), (e^{i\hat{\theta}}, b) \right) = b.$$

We see that $\pi^{-1}(b)$ consists of two free circles $e^{i\theta}$. Hence

$$\pi^{-1}(b) = S^1 \times S^1$$

and M is a T^2 -fibration over $\mathbb{CP}^1$. □