

Problem Set 3: Lie Groups and Vector Fields

Problem 1. Let

$$m : GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$$

be matrix multiplication: $m(A, B) = AB$. Let $A, B \in GL(n, \mathbb{R})$ and $\dot{A}, \dot{B} \in \text{Mat}_{n \times n}(\mathbb{R})$. Compute

$$dm|_{(A,B)}(\dot{A}, \dot{B}).$$

Let

$$i : GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$$

be matrix inversion: $i(A) = A^{-1}$. Compute $di|_A(\dot{A})$.

Problem 2. Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $g \in G$, let $C_g : G \rightarrow G$, defined by

$$C_g(h) = ghg^{-1},$$

denote the conjugacy map.

Recall that a Lie group representation on a vector space V is a smooth map $\rho : G \rightarrow GL(V)$ with the property that $\rho(e) = I_n$ and $\rho(g_1g_2) = \rho(g_1)\rho(g_2)$.

(a) Show that $\text{Ad} : G \rightarrow GL(\mathfrak{g})$ defined by $\text{Ad}_g(x) = (dC_g)_e(x)$ is a Lie group representation.

(b) Show that in the case of $G = GL(n, \mathbb{R})$, the adjoint representation is given by matrix conjugation.

(c) For $x \in \mathfrak{g}$, let $\text{ad}_x : \mathfrak{g} \rightarrow \mathfrak{g}$ be defined by $\text{ad}_x = d(\text{Ad})_e(x)$. For $G = GL(n, \mathbb{R})$, show that

$$\text{ad}_x(y) = [x, y] = xy - yx.$$

Problem 3. This problem studies $SU(2)$ as a double cover of $SO(3)$.

(a) Let σ_i denote the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Consider the map $\text{Ad} : SU(2) \rightarrow GL(\mathfrak{su}(2))$

$$\text{Ad}_g(x) = gxg^{-1}.$$

Identify $\mathfrak{su}(2)$ with $\mathbb{R}^3$ by identifying the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ with the standard basis e_1, e_2, e_3 . Show that the map φ which maps $g \in SU(2)$ to the matrix of Ad_g in the basis $\sigma_1, \sigma_2, \sigma_3$ is a homomorphism of Lie groups

$$\varphi : SU(2) \rightarrow SO(3).$$

As a first step, show that

$$\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} \cdot \begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix} = \frac{1}{2} \text{Tr } xy$$

where $x = x^i \sigma_i$ and $y = y^i \sigma_i$.

(b) Show that the path

$$\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}$$

in $SU(2)$, corresponds under φ , to the path

$$\begin{bmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in $SO(3)$. We see that a loop in $SU(2)$ corresponds to two full rotations in $SO(3)$. In fact, this calculation generalizes to the identity

$$\varphi(\exp(-i\theta\sigma_k)) = R_{e_k}(2\theta), \quad k = 1, 2, 3$$

where $R_{e_k}(\psi) \in SO(3)$ denotes rotation by angle ψ about the e_k -axis.

(c) Compute the matrices $E_\alpha \in M_{3 \times 3}(\mathbb{R})$ such that

$$d\varphi_e(\sigma_\alpha/2i) = E_\alpha, \quad \alpha = 1, 2, 3.$$

Conclude that $d\varphi_e : \mathfrak{su}(2) \rightarrow \mathfrak{so}(3)$ is an isomorphism.

(d) Denote an element $g \in SU(2)$ by

$$g = \begin{bmatrix} u & -\bar{v} \\ v & \bar{u} \end{bmatrix}, \quad u, v \in \mathbb{C}, \quad |u|^2 + |v|^2 = 1.$$

Compute the matrix $\varphi(g) \in SO(3)$.

(e) Show that $\varphi : SU(2) \rightarrow SO(3)$ is a double cover. To do this, you can first compute the kernel

$$\ker \varphi = \{g \in SU(2) : \varphi(g) = I_{3 \times 3}\},$$

and then deduce that φ is 2 : 1. Next, to show surjectivity of φ , write a matrix $A \in SO(3)$ as a product of rotations using Euler angles

$$A = R_{e_1}(\alpha)R_{e_2}(\beta)R_{e_3}(\gamma),$$

where R_{e_i} denotes a rotation about the e_i -axis, and apply part (b).

Problem 4. Let X be the vector field on

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

defined by

$$X = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

Sketch this vector field and compute the coordinate expression for X using the stereographic projection. Recall that these coordinates are setup as

$$S^1 = U \cup \tilde{U},$$

with $U = S^1 \setminus \{N\}$, $\tilde{U} = S^1 \setminus \{S\}$, and

$$\begin{aligned} \varphi : U &\rightarrow \mathbb{R}, & u &= \varphi(x, y) = \frac{x}{1-y} \\ \tilde{\varphi} : \tilde{U} &\rightarrow \mathbb{R}, & \tilde{u} &= \tilde{\varphi}(x, y) = \frac{x}{1+y}. \end{aligned}$$

The change of coordinates is $\tilde{u} = 1/u$. The problem is then to compute the local expressions

$$X|_U = X(u) \frac{\partial}{\partial u}, \quad X|_{\tilde{U}} = \tilde{X}(\tilde{u}) \frac{\partial}{\partial \tilde{u}}.$$

Problem 5. Let (x, y) be Euclidean coordinates and (r, θ) be polar coordinates on $\mathbb{R}^2 \setminus C$, where C is the negative real axis.

$$C = \{(x, y) \in \mathbb{R}^2 : x \leq 0, y = 0\}.$$

Write the vector fields

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}$$

in polar coordinates (r, θ) .

Problem 6. Find a flow on T^2 corresponding to the vector field

$$X = a \frac{\partial}{\partial \theta^1} + b \frac{\partial}{\partial \theta^2}$$

where a, b are constants. Here $T^2 = S^1 \times S^1$ and we write

$$S^1 = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}.$$

Problem 7. Consider the family of maps

$$\hat{\theta}_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix} : \mathbb{R}^3 \rightarrow \mathbb{R}^3,$$

for each $t \in \mathbb{R}$.

- (a) Prove that $\hat{\theta}_t$ induces diffeomorphisms $\theta_t : S^2 \rightarrow S^2$.
(b) Compute the vector field X associated to the flow θ_t and find the points where X is zero. To describe X , use the description

$$TS^2 = \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x| = 1, \quad x \cdot v = 0\},$$

instead of local coordinate descriptions. In other words, write your answer as

$$X = \begin{bmatrix} V_1(x, y, z) \\ V_2(x, y, z) \\ V_3(x, y, z) \end{bmatrix},$$

and find the functions V_1, V_2, V_3 .
