

Problem Set 3: Lie Groups and Vector Fields

Problem 1. Let

$$m : GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$$

be matrix multiplication: $m(A, B) = AB$. Let $A, B \in GL(n, \mathbb{R})$ and $\dot{A}, \dot{B} \in \text{Mat}_{n \times n}(\mathbb{R})$. Compute

$$dm|_{(A,B)}(\dot{A}, \dot{B}).$$

Let

$$i : GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$$

be matrix inversion: $i(A) = A^{-1}$. Compute $di|_A(\dot{A})$.

Solution. To compute the pushforward, we use the curves

$$A(t) = A + t\dot{A}, \quad B(t) = B + t\dot{B}$$

with $t \in (-\epsilon, \epsilon)$ and note that $A(t), B(t) \in GL(n, \mathbb{R})$ for small $\epsilon > 0$. Then

$$dm|_{(A,B)}(\dot{A}, \dot{B}) = \left. \frac{d}{dt} \right|_{t=0} A(t)B(t)$$

which amounts to

$$dm|_{(A,B)}(\dot{A}, \dot{B}) = \dot{A}B + A\dot{B}.$$

As for matrix inversion,

$$di|_A(\dot{A}) = \left. \frac{d}{dt} \right|_{t=0} A(t)^{-1}.$$

We must compute the derivative of $A^{-1}(t)$. Differentiating $A(t)A(t)^{-1} = I$ gives

$$\dot{A}A^{-1} + A \left. \frac{d}{dt} \right|_{t=0} A(t)^{-1} = 0.$$

Therefore

$$\left. \frac{d}{dt} \right|_{t=0} A(t)^{-1} = -A^{-1}\dot{A}A^{-1}$$

and hence

$$di|_A(\dot{A}) = -A^{-1}\dot{A}A^{-1}.$$

□

Problem 2. Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $g \in G$, let $C_g : G \rightarrow G$, defined by

$$C_g(h) = ghg^{-1},$$

denote the conjugacy map.

Recall that a Lie group representation on a vector space V is a smooth map $\rho : G \rightarrow GL(V)$ with the property that $\rho(e) = I_n$ and $\rho(g_1g_2) = \rho(g_1)\rho(g_2)$.

(a) Show that $\text{Ad} : G \rightarrow GL(\mathfrak{g})$ defined by $\text{Ad}_g(x) = (dC_g)_e(x)$ is a Lie group representation.

(b) Show that in the case of $G = GL(n, \mathbb{R})$, the adjoint representation is given by matrix conjugation.

(c) For $x \in \mathfrak{g}$, let $\text{ad}_x : \mathfrak{g} \rightarrow \mathfrak{g}$ be defined by $\text{ad}_x = d(\text{Ad})_e(x)$. For $G = GL(n, \mathbb{R})$, show that

$$\text{ad}_x(y) = [x, y] = xy - yx.$$

Solution. To show Ad is a representation, we need to show $\text{Ad}_{gh} = \text{Ad}_g\text{Ad}_h$. Let $V \in \mathfrak{g}$, meaning that V is a tangent vector at the identity $e \in G$. By definition of the pushforward, we have

$$\text{Ad}_{gh}V = \left. \frac{d}{dt} \right|_{t=0} C_{gh} \circ \gamma = \left. \frac{d}{dt} \right|_{t=0} gh\gamma(t)h^{-1}g^{-1}.$$

Here γ is a path with $\gamma(0) = e$ and $\dot{\gamma} = V$. Next, we have

$$\text{Ad}_hV = \left. \frac{d}{dt} \right|_{t=0} h\gamma(t)h^{-1}.$$

The path $\tilde{\gamma} = h\gamma h^{-1}$ satisfies $\tilde{\gamma}(0) = e$ and $\dot{\tilde{\gamma}}(0) = \text{Ad}_hV$. By definition of the pushforward,

$$\text{Ad}_g\text{Ad}_hV = \left. \frac{d}{dt} \right|_{t=0} C_g \circ \tilde{\gamma} = \left. \frac{d}{dt} \right|_{t=0} gh\gamma(t)h^{-1}g^{-1}.$$

This is equal to $\text{Ad}_{gh}V$, proving part (a).

(b) For matrix groups $G = GL(n, \mathbb{R})$, let $g \in G$, $x \in \mathfrak{g}$ and we need to show $\text{Ad}_g(x) = gxg^{-1}$. Consider the path

$$\gamma(t) = e^{tx}, \quad \gamma(0) = e, \quad \gamma'(0) = x.$$

We have

$$\text{Ad}_g(x) = \left. \frac{d}{dt} \right|_{t=0} C_g \circ \gamma(t) = \left. \frac{d}{dt} \right|_{t=0} ge^{tx}g^{-1}.$$

Then

$$\text{Ad}_g(x) = \left. \frac{d}{dt} \right|_{t=0} g \sum_{k=0}^{\infty} \frac{(tx)^k}{k!} g^{-1} = gxg^{-1},$$

as desired.

(c) Next, we have $(d\text{Ad})_e : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ and we need to show $(d\text{Ad})_e(x) \in \text{End}(\mathfrak{g})$ is the matrix commutator. By definition

$$d(\text{Ad})_e(x) = \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\gamma(t)}, \quad \gamma(t) = e^{tx}.$$

Therefore

$$d(\text{Ad})_e(x)(y) = \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\gamma(t)}(y),$$

and so by part (b) we have

$$d(\text{Ad})_e(x)(y) = \left. \frac{d}{dt} \right|_{t=0} e^{tx} y e^{-tx} = xy - yx.$$

□

Problem 3. This problem studies $SU(2)$ as a double cover of $SO(3)$.

(a) Let σ_i denote the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Consider the map $\text{Ad} : SU(2) \rightarrow GL(i\mathfrak{su}(2))$

$$\text{Ad}_g(x) = gxg^{-1}.$$

Identify $i\mathfrak{su}(2)$ with $\mathbb{R}^3$ by identifying the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ with the standard basis e_1, e_2, e_3 . Show that the map φ which maps $g \in SU(2)$ to the matrix of Ad_g in the basis $\sigma_1, \sigma_2, \sigma_3$ is a homomorphism of Lie groups

$$\varphi : SU(2) \rightarrow SO(3).$$

As a first step, show that

$$\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} \cdot \begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix} = \frac{1}{2} \text{Tr } xy$$

where $x = x^i \sigma_i$ and $y = y^i \sigma_i$.

(b) Show that the path

$$\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}$$

in $SU(2)$, corresponds under φ , to the path

$$\begin{bmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in $SO(3)$. We see that a loop in $SU(2)$ corresponds to two full rotations in $SO(3)$. In fact, this calculation generalizes to the identity

$$\varphi(\exp(-i\theta\sigma_k)) = R_{e_k}(2\theta), \quad k = 1, 2, 3$$

where $R_{e_k}(\psi) \in SO(3)$ denotes rotation by angle ψ about the e_k -axis.

(c) Compute the matrices $E_\alpha \in M_{3 \times 3}(\mathbb{R})$ such that

$$d\varphi_e(\sigma_\alpha/2i) = E_\alpha, \quad \alpha = 1, 2, 3.$$

Conclude that $d\varphi_e : \mathfrak{su}(2) \rightarrow \mathfrak{so}(3)$ is an isomorphism.

(d) Denote an element $g \in SU(2)$ by

$$g = \begin{bmatrix} u & -\bar{v} \\ v & \bar{u} \end{bmatrix}, \quad u, v \in \mathbb{C}, \quad |u|^2 + |v|^2 = 1.$$

Compute the matrix $\varphi(g) \in SO(3)$.

(e) Show that $\varphi : SU(2) \rightarrow SO(3)$ is a double cover. To do this, you can first compute the kernel

$$\ker \varphi = \{g \in SU(2) : \varphi(g) = I_{3 \times 3}\},$$

and then deduce that φ is 2 : 1. Next, to show surjectivity of φ , write a matrix $A \in SO(3)$ as a product of rotations using Euler angles

$$A = R_{e_1}(\alpha)R_{e_2}(\beta)R_{e_3}(\gamma),$$

where R_{e_i} denotes a rotation about the e_i -axis, and apply part (b).

Solution. We start with part (a). There are two steps here. The first is to show that $\varphi(g)$ is a matrix in $SO(3)$. The second is to show that $\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2)$. The second step was showed in the previous problem since it is equivalent to

$$\text{Ad}_{gh} = \text{Ad}_g \text{Ad}_h.$$

So what remains to be shown is that if $g \in SU(2)$, then $\varphi(g) \in SO(3)$. The matrix $\varphi(g)$ has columns

$$\varphi(g) = \begin{bmatrix} (g\sigma_1g^{-1})^1 & (g\sigma_2g^{-1})^1 & (g\sigma_3g^{-1})^1 \\ (g\sigma_1g^{-1})^2 & (g\sigma_2g^{-1})^2 & (g\sigma_3g^{-1})^2 \\ (g\sigma_1g^{-1})^3 & (g\sigma_2g^{-1})^3 & (g\sigma_3g^{-1})^3 \end{bmatrix},$$

where we write

$$(g\sigma_1g^{-1})^i = (g\sigma_1g^{-1})^i \sigma_i.$$

Are these columns orthonormal? Assuming the hint, we have

$$\begin{bmatrix} (g\sigma_1g^{-1})^1 \\ (g\sigma_1g^{-1})^2 \\ (g\sigma_1g^{-1})^3 \end{bmatrix} \cdot \begin{bmatrix} (g\sigma_2g^{-1})^1 \\ (g\sigma_2g^{-1})^2 \\ (g\sigma_2g^{-1})^3 \end{bmatrix} = \frac{1}{2} \text{Tr}(g\sigma_1g^{-1})(g\sigma_2g^{-1}) = \frac{1}{2} \text{Tr} \sigma_1\sigma_2 = 0.$$

A similar calculation shows that all columns are orthonormal. Therefore, if we can prove the hint, then we can conclude that $\varphi(g) \in O(3)$. The hint simply follows from the Pauli matrix identities

$$\text{Tr} \sigma_i \sigma_j = 0, \quad i \neq j, \quad \text{Tr} \sigma_i^2 = 2,$$

which implies

$$\text{Tr} xy = x^i y^j \text{Tr} \sigma_i \sigma_j = 2x^i y^i.$$

Therefore we have proved that φ maps into $O(3)$. So the determinant can be ± 1 . But the function $\det \varphi : SU(2) \rightarrow \{\pm 1\}$ is continuous and $SU(2)$ is connected, so the determinant is always 1 since $\det \varphi(I) = 1$. Thus φ maps into $SO(3)$. Here is why $SU(2)$ is connected: as noted in the statement of part (d), an element $g \in SU(2)$ is in correspondence with

$$S^3 = \{u, v \in \mathbb{C} : |u|^2 + |v|^2 = 1\}.$$

Since S^3 is connected, then so is $SU(2)$.

(b) Let's compute

$$\varphi\left(\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}\right).$$

We first act on the first basis element σ_1 .

$$\text{Ad}\left(\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}\right)(\sigma_1) = \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} = \begin{bmatrix} 0 & e^{-2i\theta} \\ e^{2i\theta} & 0 \end{bmatrix}.$$

Therefore

$$\text{Ad}\left(\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}\right)(\sigma_1) = \cos 2\theta \sigma_1 + \sin 2\theta \sigma_2.$$

Next, we act on the second basis element σ_2 .

$$\text{Ad}\left(\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}\right)(\sigma_2) = \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} = \begin{bmatrix} 0 & -ie^{-2i\theta} \\ ie^{2i\theta} & 0 \end{bmatrix}$$

Therefore

$$\text{Ad}\left(\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}\right)(\sigma_2) = -\sin 2\theta \sigma_1 + \cos 2\theta \sigma_2.$$

Finally, we compute

$$\text{Ad}\left(\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}\right)(\sigma_3) = \sigma_3.$$

We can now write

$$\text{Ad}\left(\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}\right)$$

as a 3×3 matrix by using the basis $\sigma_1, \sigma_2, \sigma_3$. The result is

$$\varphi\left(\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}\right) = \begin{bmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) We computed in the previous problem that

$$d\varphi_e(\sigma_1/2i)(x^i \sigma_i) = \frac{1}{2i}[\sigma_1, x^i \sigma_i] = x^2 \sigma_3 - x^3 \sigma_2.$$

Here we used the well-known identities

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k.$$

Therefore

$$d\varphi_e(\sigma_1/2i) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We omit the calculations of the pushforward of σ_2 and σ_3 , and give the final answer:

$$d\varphi_e(\sigma_\alpha/2i) = E_\alpha,$$

where

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

are the generators of rotation.

(d) Let

$$g = \begin{bmatrix} u & -\bar{v} \\ v & \bar{u} \end{bmatrix}, \quad g^{-1} = \begin{bmatrix} \bar{u} & \bar{v} \\ -v & u \end{bmatrix}.$$

We directly compute

$$g\sigma_1g^{-1} = \begin{bmatrix} -uv - \bar{u}\bar{v} & u^2 - \bar{v}^2 \\ \bar{u}^2 - v^2 & uv + \bar{u}\bar{v} \end{bmatrix},$$

which is

$$g\sigma_1g^{-1} = \operatorname{Re}(u^2 - \bar{v}^2)\sigma_1 - \operatorname{Im}(u^2 - \bar{v}^2)\sigma_2 + (-uv - \bar{u}\bar{v})\sigma_3.$$

We repeat for basis element σ_2 . This gives

$$g\sigma_2g^{-1} = \begin{bmatrix} -i\bar{u}\bar{v} + iuv & -iu^2 - i\bar{v}^2 \\ i\bar{u}^2 + iv^2 & i\bar{u}\bar{v} - iuv \end{bmatrix},$$

and

$$g\sigma_2g^{-1} = -\operatorname{Re}(iu^2 + i\bar{v}^2)\sigma_1 + \operatorname{Im}(iu^2 + i\bar{v}^2)\sigma_2 + (iuv - i\bar{u}\bar{v})\sigma_3$$

Lastly, we repeat for basis element σ_3 , and obtain

$$g\sigma_3g^{-1} = \begin{bmatrix} |u|^2 - |v|^2 & 2u\bar{v} \\ 2\bar{u}v & |v|^2 - |u|^2 \end{bmatrix},$$

which is

$$g\sigma_3g^{-1} = \operatorname{Re}(2u\bar{v})\sigma_1 - \operatorname{Im}(2u\bar{v})\sigma_2 + (|u|^2 - |v|^2)\sigma_3.$$

Putting all the $g\sigma_i g^{-1}$ together, we obtain

$$\varphi(g) = \begin{bmatrix} \operatorname{Re}(u^2 - \bar{v}^2) & -\operatorname{Re}(iu^2 + i\bar{v}^2) & 2\operatorname{Re}(u\bar{v}) \\ -\operatorname{Im}(u^2 - \bar{v}^2) & \operatorname{Im}(iu^2 + i\bar{v}^2) & -2\operatorname{Im}(u\bar{v}) \\ -2\operatorname{Re}(uv) & 2\operatorname{Re}(iuv) & |u|^2 - |v|^2 \end{bmatrix},$$

which we know from part (a) is a matrix in $SO(3)$.

(e) We see that if $\varphi(g) = I$, then $u\bar{v} = 0$ so either u or v is zero. But also $|u|^2 - |v|^2 = 1$, so actually $v = 0$. Since

$$\operatorname{Re} u^2 = 1, \quad \operatorname{Im} u^2 = 0,$$

we see that u is either 1 or -1 . Therefore

$$\operatorname{Ker} \varphi = \{I, -I\}.$$

Since φ is a group homomorphism, the mapping is $2 : 1$. Next, we show φ is surjective. Let $A \in SO(3)$ and write

$$A = R_{e_1}(\alpha)R_{e_2}(\beta)R_{e_3}(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix} \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$g = \exp\left(-\frac{i\alpha}{2}\sigma_1\right) \exp\left(-\frac{i\beta}{2}\sigma_2\right) \exp\left(-\frac{i\gamma}{2}\sigma_3\right)$$

solves $\varphi(g) = A$ by the homomorphism property and part (b). Let us end by explaining the decomposition of $A \in SO(3)$ into a product of rotations about the x , y , z axes. Consider a unit vector Ae_3 . Any unit vector can be written as a product $R_{e_1}(\alpha)R_{e_2}(\beta)e_3$. So we can write

$$Ae_3 = R_{e_1}(\alpha)R_{e_2}(\beta)e_3$$

Then $(R_{e_1}(\alpha)R_{e_2}(\beta))^{-1}A$ is a rotation fixing e_3 , hence is equal to $R_{e_3}(\gamma)$ for some angle γ . This proves $A = R_{e_1}(\alpha)R_{e_2}(\beta)R_{e_3}(\gamma)$. $\square$

Problem 4. Let X be the vector field on

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

defined by

$$X = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

Sketch this vector field and compute the coordinate expression for X using the stereographic projection. Recall that these coordinates are setup as

$$S^1 = U \cup \tilde{U},$$

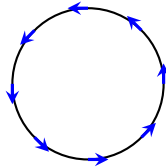
with $U = S^1 \setminus \{N\}$, $\tilde{U} = S^1 \setminus \{S\}$, and

$$\begin{aligned} \varphi : U &\rightarrow \mathbb{R}, & u &= \varphi(x, y) = \frac{x}{1-y} \\ \tilde{\varphi} : \tilde{U} &\rightarrow \mathbb{R}, & \tilde{u} &= \tilde{\varphi}(x, y) = \frac{x}{1+y}. \end{aligned}$$

The change of coordinates is $\tilde{u} = 1/u$. The problem is then to compute the local expressions

$$X|_U = X(u) \frac{\partial}{\partial u}, \quad X|_{\tilde{U}} = \tilde{X}(\tilde{u}) \frac{\partial}{\partial \tilde{u}}.$$

Solution. This vector field flows counterclockwise around the circle.



We note that

$$X = \frac{d}{dt}\gamma(t), \quad \gamma(t) = (x(t), y(t)) = (\cos t, \sin t).$$

In local coordinates (U, u) , the path $\gamma(t)$ appears as

$$\gamma|_U(t) = \frac{\cos t}{1 - \sin t}.$$

Therefore

$$X|_U = \frac{d}{dt} \left[\frac{\cos t}{1 - \sin t} \right] \frac{\partial}{\partial u}.$$

This works out to be

$$X|_U = \frac{1}{1 - y} \frac{\partial}{\partial u},$$

but we need to convert to u coordinates via

$$(x, y) = \varphi^{-1}(u) = \left(\frac{2u}{1 + u^2}, \frac{u^2 - 1}{1 + u^2} \right).$$

Thus

$$X|_U = \frac{1}{2}(1 + u^2) \frac{\partial}{\partial u}.$$

To compute $X|_{\tilde{U}}$, we use the vector field transformation law $\tilde{X} = \frac{\partial \tilde{u}}{\partial u} X$. This gives

$$X|_{\tilde{U}} = -\frac{1}{2}(1 + \tilde{u}^2) \frac{\partial}{\partial \tilde{u}},$$

over the chart $(\tilde{U}, \tilde{u})$. □

Problem 5. Let (x, y) be Euclidean coordinates and (r, θ) be polar coordinates on $\mathbb{R}^2 \setminus C$, where C is the negative real axis.

$$C = \{(x, y) \in \mathbb{R}^2 : x \leq 0, y = 0\}.$$

Write the vector fields

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}$$

in polar coordinates (r, θ) .

Solution. The transformation law is

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta}.$$

First, we find $\frac{\partial r}{\partial x}$. Since $r = \sqrt{x^2 + y^2}$, we have

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}.$$

In polar coordinates $x = r \cos \theta$, $y = r \sin \theta$, this is

$$\frac{\partial r}{\partial x} = \cos \theta.$$

To find $\frac{\partial \theta}{\partial x}$, we take $\frac{\partial}{\partial x}$ of $x = r \cos \theta$ and $y = r \sin \theta$.

$$1 = \frac{\partial r}{\partial x} \cos \theta - r \sin \theta \frac{\partial \theta}{\partial x}, \quad (1)$$

$$0 = \frac{\partial r}{\partial x} \sin \theta + r \cos \theta \frac{\partial \theta}{\partial x}. \quad (2)$$

Multiply (1) by $\sin \theta$ and multiply (2) by $\cos \theta$, and subtract them

$$\sin \theta = \left[-r \sin^2 \theta - r \cos^2 \theta \right] \frac{\partial \theta}{\partial x}.$$

Therefore

$$\frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{r}.$$

Combining $\frac{\partial r}{\partial x}$ and $\frac{\partial \theta}{\partial x}$ gives

$$\frac{\partial}{\partial x} = \left[\cos \theta \right] \frac{\partial}{\partial r} + \left[-\frac{\sin \theta}{r} \right] \frac{\partial}{\partial \theta}.$$

Next, we compute $\frac{\partial}{\partial y}$ in polar coordinates. First, we will need

$$\frac{\partial r}{\partial y} = \sin \theta.$$

To find $\frac{\partial \theta}{\partial y}$, we take $\frac{\partial}{\partial y}$ of $x = r \cos \theta$ and $y = r \sin \theta$.

$$0 = \frac{\partial r}{\partial y} \cos \theta - r \sin \theta \frac{\partial \theta}{\partial y},$$

$$1 = \frac{\partial r}{\partial y} \sin \theta + r \cos \theta \frac{\partial \theta}{\partial y}.$$

The same strategy as last time gives

$$\frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r}.$$

Therefore

$$\frac{\partial}{\partial y} = \left[\sin \theta \right] \frac{\partial}{\partial r} + \left[\frac{\cos \theta}{r} \right] \frac{\partial}{\partial \theta}.$$

□

Problem 6. Find a flow on T^2 corresponding to the vector field

$$X = a \frac{\partial}{\partial \theta^1} + b \frac{\partial}{\partial \theta^2}$$

where a, b are constants. Here $T^2 = S^1 \times S^1$ and we write

$$S^1 = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}.$$

Solution. The flow $\theta_t : T^2 \rightarrow T^2$ is

$$\theta_t(e^{i\theta^1}, e^{i\theta^2}) = (e^{i(\theta^1+at)}, e^{i(\theta^2+bt)})$$

Indeed, in local angle coordinates over an open set U , then

$$\theta_t|_U(\theta^1, \theta^2) = (\theta^1 + at, \theta^2 + bt)$$

and so

$$X|_U = \frac{d}{dt} \Big|_{t=0} \theta_t|_U = (a, b)$$

which in the vector field notation is

$$X|_U = a \frac{\partial}{\partial \theta^1} + b \frac{\partial}{\partial \theta^2}.$$

□

Problem 7. Consider the family of maps

$$\hat{\theta}_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix} : \mathbb{R}^3 \rightarrow \mathbb{R}^3,$$

for each $t \in \mathbb{R}$.

(a) Prove that $\hat{\theta}_t$ induces diffeomorphisms $\theta_t : S^2 \rightarrow S^2$.

(b) Compute the vector field X associated to the flow θ_t and find the points where X is zero. To describe X , use the description

$$TS^2 = \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x| = 1, \quad x \cdot v = 0\},$$

instead of local coordinate descriptions. In other words, write your answer as

$$X = \begin{bmatrix} V_1(x, y, z) \\ V_2(x, y, z) \\ V_3(x, y, z) \end{bmatrix},$$

and find the functions V_1, V_2, V_3 .

Solution. Note that if

$$\left| \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right| = 1,$$

then

$$\left| \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right| = 1,$$

so $\theta_t : S^2 \rightarrow S^2$ is well-defined. The inverse of θ_t is rotation by the opposite angle:

$$\theta_t^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}.$$

Expanded in full, these maps are

$$\theta_t(x, y, z) = (x, \cos ty + \sin tz, -\sin ty + \cos tz),$$

$$\theta_t^{-1}(x, y, z) = (x, \cos ty - \sin tz, +\sin ty + \cos tz).$$

These are smooth, since as we know from Problem Set 2, if $\tilde{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a smooth map with $\tilde{F}(S^2) \subseteq S^2$, then $\tilde{F}|_{S^2} : S^2 \rightarrow S^2$ is a smooth map. To obtain the generating vector field, we compute

$$X = \left. \frac{d}{dt} \right|_{t=0} \theta_t = \begin{bmatrix} 0 \\ z \\ -y \end{bmatrix}.$$

This vector field only vanishes at $(\pm 1, 0, 0)$.

□