

Problem Set 4: Vector Bundles

Problem 1. Recall that in a previous problem set, we covered $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$ by the open sets

$$U_i = \{[x_0, \dots, x_n] : x_i \neq 0\},$$

and equipped each set with local coordinates. Let

$$L = \{([x], w) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : w \in [x]\}.$$

Define the projection

$$\pi : L \rightarrow \mathbb{RP}^n$$

by $\pi([x], w) = [x]$. We can view L as a line bundle by trivializing L over U_i by

$$\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}, \quad \varphi_i([x], w) = ([x], w_i).$$

Here we write $w = (w_0, w_1, \dots, w_n)$.

(a) Compute the transition function

$$\tau_{ij} : U_i \cap U_j \rightarrow \mathbb{R}^*.$$

Recall our conventions $\varphi_i \circ \varphi_j^{-1}(x, v) = (x, \tau_{ij}v)$.

(b) Let $v \in \mathbb{R}^{n+1}$. Such a vector defines a section $\sigma \in \Gamma(L)$ by

$$\sigma[x] = \left([x], \text{proj}_{[x]}(v) \right).$$

For concreteness, let us just take $n = 1$ and $v = (1, 0)$. Using the conventions $(\varphi_i \circ \sigma)(x) = (x, s_i(x))$, compute s_0 and s_1 and verify $s_0 = \tau_{01}s_1$. Use the following notation: the local U_0 -coordinate is denoted $u = x_1/x_0$ and the local U_1 -coordinate is denoted $\tilde{u} = x_0/x_1$. The local section $s_0(u)$ should be written in terms of its own local coordinate, and similarly for $s_1(\tilde{u})$.

Problem 2. Let $E \subset S^1 \times \mathbb{R}^2$ be the set of points $(e^{i\theta}, v_1, v_2)$ such that

$$\begin{aligned}(\cos \theta)v_1 + (\sin \theta)v_2 &= v_1 \\ (\sin \theta)v_1 - (\cos \theta)v_2 &= v_2.\end{aligned}$$

This space forms the Möbius bundle $E \rightarrow S^1$ with projection $\pi(e^{i\theta}, v_1, v_2) = e^{i\theta}$. We will use the charts over $S^1 = U \cup \tilde{U}$ given by

$$U = \{e^{i\theta} : \theta \in (0, 2\pi)\}, \quad \tilde{U} = \{e^{i\tilde{\theta}} : \tilde{\theta} \in (-\pi, \pi)\}$$

with coordinates (U, θ) and $(\tilde{U}, \tilde{\theta})$.

(a) Show that when $0 < \theta < 2\pi$ and $(e^{i\theta}, v_1, v_2) \in E$, we can write

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = r \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix}$$

with

$$r = \frac{v_2}{\sin(\theta/2)}.$$

Similarly, when $-\pi < \tilde{\theta} < \pi$ and $(e^{i\tilde{\theta}}, v_1, v_2) \in E$, we can write

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \rho \begin{bmatrix} \cos(\tilde{\theta}/2) \\ \sin(\tilde{\theta}/2) \end{bmatrix}$$

with

$$\rho = \frac{v_1}{\cos(\tilde{\theta}/2)}.$$

We see that when the angle turns $0 \rightsquigarrow 2\pi$ around the circle, the vector flips direction.

(b) Give E the structure of a vector bundle over S^1 by defining a trivialization over U by

$$\left(e^{i\theta}, r \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix} \right) \mapsto (\theta, r),$$

and a trivialization over $\tilde{U}$ by

$$\left(e^{i\tilde{\theta}}, \rho \begin{bmatrix} \cos(\tilde{\theta}/2) \\ \sin(\tilde{\theta}/2) \end{bmatrix} \right) \mapsto (\tilde{\theta}, \rho).$$

Compute the transition function $\tau_{12} : U \cap \tilde{U} \rightarrow \mathbb{R}^*$.

(c) Show that

$$\sigma(e^{i\theta}) = (e^{i\theta}, \sin \theta, 1 - \cos \theta) \in E$$

defines a smooth section $\sigma \in \Gamma(E)$ by finding the local representatives over the trivializations $\pi^{-1}(U)$ and $\pi^{-1}(\tilde{U})$. Write

$$\sigma|_U = (\theta, s(\theta)), \quad \sigma|_{\tilde{U}} = (\tilde{\theta}, \tilde{s}(\tilde{\theta})),$$

and verify $\tilde{s} = \tau_{12}s$.

Problem 3. Let $E \rightarrow M$ and $\tilde{E} \rightarrow M$ be two vector bundles. Suppose E and $\tilde{E}$ can be trivialized by the same cover $M = \bigcup U_\alpha$. Let

$$\tau_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{R}), \quad \tilde{\tau}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{R}),$$

denote the transition functions.

(a) Show that E is isomorphic to $\tilde{E}$ if and only if there exists local functions $h_\alpha : U_\alpha \rightarrow GL(k, \mathbb{R})$ such that

$$\tilde{\tau}_{\alpha\beta} = h_\alpha \tau_{\alpha\beta} h_\beta^{-1},$$

on $U_\alpha \cap U_\beta$.

(b) Show that a line bundle $L \rightarrow M$ is isomorphic to the trivial bundle $M \times \mathbb{R}$ if and only if it admits a nowhere vanishing section.

Problem 4. In this problem, we will show that the direct sum of two copies of the Möbius bundle is isomorphic to a trivial bundle. To do this, you may use the cover $S^1 = U_1 \cup U_2$ with

$$U_1 = \{e^{i\theta} : \theta \in (0, 2\pi)\}, \quad U_2 = \{e^{i\tilde{\theta}} : \tilde{\theta} \in (-\pi, \pi)\}$$

with change of coordinates

$$\tilde{\theta} = \begin{cases} \theta, & \text{if } 0 < \theta < \pi \\ \theta - 2\pi, & \text{if } \pi < \theta < 2\pi. \end{cases}$$

As we know from the previous problem, the Möbius bundle $E \rightarrow S^1$ is defined by the data $(U_1 \cap U_2, \tau_{12})$ with

$$\tau_{12} = \begin{cases} 1, & \text{if } 0 < \theta < \pi \\ -1, & \text{if } \pi < \theta < 2\pi. \end{cases}$$

To show $E \oplus E \rightarrow S^1$ is the trivial bundle, find

$$h_\alpha : U_\alpha \rightarrow GL(2, \mathbb{R})$$

such that on $U_1 \cap U_2$ there holds

$$I_{2 \times 2} = h_1 \begin{bmatrix} \tau_{12} & 0 \\ 0 & \tau_{12} \end{bmatrix} h_2^{-1}.$$

Hint: the h_α will be of the form

$$\begin{bmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{bmatrix},$$

where $\psi(\theta)$ on U_1 and $\psi(\tilde{\theta})$ on U_2 .

Problem 5. Let $\pi : E \rightarrow M$ be a vector bundle and let $s : M \rightarrow E$ be a section. Show that

$$S = \{s(p) : p \in M\}$$

is a submanifold of E which is diffeomorphic to M .
