

Problem Set 4: Vector Bundles

Problem 1. Recall that in a previous problem set, we covered $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$ by the open sets

$$U_i = \{[x_0, \dots, x_n] : x_i \neq 0\},$$

and equipped each set with local coordinates. Let

$$L = \{([x], w) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : w \in [x]\}.$$

Define the projection

$$\pi : L \rightarrow \mathbb{RP}^n$$

by $\pi([x], w) = [x]$. We can view L as a line bundle by trivializing L over U_i by

$$\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}, \quad \varphi_i([x], w) = ([x], w_i).$$

Here we write $w = (w_0, w_1, \dots, w_n)$.

(a) Compute the transition function

$$\tau_{ij} : U_i \cap U_j \rightarrow \mathbb{R}^*.$$

Recall our conventions $\varphi_i \circ \varphi_j^{-1}(x, v) = (x, \tau_{ij}v)$.

(b) Let $v \in \mathbb{R}^{n+1}$. Such a vector defines a section $\sigma \in \Gamma(L)$ by

$$\sigma[x] = \left([x], \text{proj}_{[x]}(v)\right).$$

For concreteness, let us just take $n = 1$ and $v = (1, 0)$. Using the conventions $(\varphi_i \circ \sigma)(x) = (x, s_i(x))$, compute s_0 and s_1 and verify $s_0 = \tau_{01}s_1$. Use the following notation: the local U_0 -coordinate is denoted $u = x_1/x_0$ and the local U_1 -coordinate is denoted $\tilde{u} = x_0/x_1$. The local section $s_0(u)$ should be written in terms of its own local coordinate, and similarly for $s_1(\tilde{u})$.

Solution. We compute $\tau_{ij} = \varphi_i \circ \varphi_j^{-1}$. We start with

$$\varphi_j^{-1}([x], \lambda) = ([x], w_\lambda), \quad w_\lambda = \left(\frac{\lambda x_0}{x_j}, \dots, \frac{\lambda x_{j-1}}{x_j}, \lambda, \frac{\lambda x_{j+1}}{x_j}, \dots, \frac{\lambda x_n}{x_j}\right).$$

Therefore

$$\varphi_i \circ \varphi_j^{-1}([x], \lambda) = \left([x], \frac{\lambda x_i}{x_j}\right),$$

and we read-off

$$\tau_{ij} = \frac{x_i}{x_j}.$$

Next, we study the section $\sigma \in \Gamma(L)$. First, we expand the orthogonal Euclidean projection $\text{proj}_{[x]} v$ of the vector $v = (1, 0)$ onto the line $[x]$.

$$\sigma[x] = \left([x_0, x_1], \frac{x_0}{x_0^2 + x_1^2} (x_0, x_1) \right).$$

To find s_0 , we compute

$$\varphi_0 \circ \sigma = \left([x_0, x_1], \frac{x_0^2}{x_0^2 + x_1^2} \right).$$

In terms of the local U_0 -coordinate $u = x_1/x_0$, we can read-off

$$s_0(u) = \frac{1}{1 + u^2}.$$

Similarly, we compute s_1 by

$$\varphi_1 \circ \sigma = \left([x_0, x_1], \frac{x_0 x_1}{x_0^2 + x_1^2} \right)$$

and in terms of the local U_1 -coordinate $\tilde{u} = x_0/x_1$, we have

$$s_1(\tilde{u}) = \frac{\tilde{u}}{1 + \tilde{u}^2}.$$

We can now verify

$$s_0 = \tau_{01} s_1,$$

which is just the equation

$$\frac{1}{1 + u^2} = \frac{1}{u} \frac{\tilde{u}}{1 + \tilde{u}^2}$$

under the relation $\tilde{u} = 1/u$. □

Problem 2. Let $E \subset S^1 \times \mathbb{R}^2$ be the set of points $(e^{i\theta}, v_1, v_2)$ such that

$$\begin{aligned}(\cos \theta)v_1 + (\sin \theta)v_2 &= v_1 \\ (\sin \theta)v_1 - (\cos \theta)v_2 &= v_2.\end{aligned}$$

This space forms the Möbius bundle $E \rightarrow S^1$ with projection $\pi(e^{i\theta}, v_1, v_2) = e^{i\theta}$. We will use the charts over $S^1 = U \cup \tilde{U}$ given by

$$U = \{e^{i\theta} : \theta \in (0, 2\pi)\}, \quad \tilde{U} = \{e^{i\tilde{\theta}} : \tilde{\theta} \in (-\pi, \pi)\}$$

with coordinates (U, θ) and $(\tilde{U}, \tilde{\theta})$.

(a) Show that when $0 < \theta < 2\pi$ and $(e^{i\theta}, v_1, v_2) \in E$, we can write

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = r \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix}$$

with

$$r = \frac{v_2}{\sin(\theta/2)}.$$

Similarly, when $-\pi < \tilde{\theta} < \pi$ and $(e^{i\tilde{\theta}}, v_1, v_2) \in E$, we can write

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \rho \begin{bmatrix} \cos(\tilde{\theta}/2) \\ \sin(\tilde{\theta}/2) \end{bmatrix}$$

with

$$\rho = \frac{v_1}{\cos(\tilde{\theta}/2)}.$$

We see that when the angle turns $0 \rightsquigarrow 2\pi$ around the circle, the vector flips direction.

(b) Give E the structure of a vector bundle over S^1 by defining a trivialization over U by

$$\left(e^{i\theta}, r \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix} \right) \mapsto (\theta, r),$$

and a trivialization over $\tilde{U}$ by

$$\left(e^{i\tilde{\theta}}, \rho \begin{bmatrix} \cos(\tilde{\theta}/2) \\ \sin(\tilde{\theta}/2) \end{bmatrix} \right) \mapsto (\tilde{\theta}, \rho).$$

Compute the transition function $\tau_{12} : U \cap \tilde{U} \rightarrow \mathbb{R}^*$.

(c) Show that

$$\sigma(e^{i\theta}) = (e^{i\theta}, \sin \theta, 1 - \cos \theta) \in E$$

defines a smooth section $\sigma \in \Gamma(E)$ by finding the local representatives over the trivializations $\pi^{-1}(U)$ and $\pi^{-1}(\tilde{U})$. Write

$$\sigma|_U = (\theta, s(\theta)), \quad \sigma|_{\tilde{U}} = (\tilde{\theta}, \tilde{s}(\tilde{\theta})),$$

and verify $\tilde{s} = \tau_{12}s$.

Solution. For (a), let $(e^{i\theta}, v_1, v_2) \in E$ with $0 < \theta < 2\pi$. We start from $(\cos \theta)v_1 + (\sin \theta)v_2 = v_1$ and solve for v_1 .

$$v_1 = \frac{\sin \theta}{1 - \cos \theta} v_2.$$

Using trig identities $\sin 2x = 2 \sin x \cos x$ and $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, this can be rewritten as

$$v_1 = \frac{\cos(\theta/2)}{\sin(\theta/2)} v_2.$$

Set $r = v_2 / \sin(\theta/2)$. Then

$$v_1 = r \cos(\theta/2), \quad v_2 = r \sin(\theta/2)$$

as required. The other formula over $-\pi < \tilde{\theta} < \pi$ is verified similarly and the verification is omitted.

For (b), the transition function is defined on $U \cap \tilde{U} = U_+ \cup U_-$, where

$$U_+ = \{e^{i\theta} : 0 < \theta < \pi\}, \quad U_- = \{e^{i\theta} : \pi < \theta < 2\pi\}.$$

On U_+ , we have $\theta = \tilde{\theta} \in (0, \pi)$ and so

$$\rho = \frac{v_1}{\cos(\tilde{\theta}/2)} = \frac{v_1}{\cos(\theta/2)} = r \implies \tau_{12}|_{U_+} = 1$$

where the last line follows from

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = r \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix}$$

together with $\theta \neq \pi$ on U_+ (otherwise division by $\cos(\theta/2)$ is forbidden). On U_- , $\tilde{\theta} = \theta - 2\pi$ and so

$$\rho = \frac{v_1}{\cos(\tilde{\theta}/2)} = \frac{v_1}{\cos(\theta/2 - \pi)} = -\frac{v_1}{\cos(\theta/2)} = -r.$$

Altogether, $\rho = \tau_{12}r$ on $U \cap \tilde{U}$ with

$$\tau_{12} = \begin{cases} 1, & \text{on } U_+, \\ -1, & \text{on } U_-. \end{cases}$$

For (c), the section $\sigma : S^1 \rightarrow E$, written in the trivialization over U , takes the local expression

$$\sigma|_U = \left(\theta, \frac{v_2(\theta)}{\sin(\theta/2)} \right)$$

and so

$$s(\theta) = \frac{1 - \cos \theta}{\sin(\theta/2)} = 2 \sin(\theta/2).$$

On the other hand, written in the trivialization over $\tilde{U}$, the local expression is

$$\sigma|_{\tilde{U}} = \left(\tilde{\theta}, \frac{v_1}{\cos(\tilde{\theta}/2)} \right)$$

and so

$$\tilde{s}(\tilde{\theta}) = \frac{\sin \tilde{\theta}}{\cos(\tilde{\theta}/2)} = 2 \sin(\tilde{\theta}/2).$$

It follows from

$$\tilde{\theta} = \begin{cases} \theta, & \text{on } U_+, \\ \theta - 2\pi, & \text{on } U_-, \end{cases}$$

that

$$\tilde{s} = \tau_{12}s$$

since $\sin((\theta - 2\pi)/2) = -\sin(\theta/2)$.

□

Problem 3. Let $E \rightarrow M$ and $\tilde{E} \rightarrow M$ be two vector bundles. Suppose E and $\tilde{E}$ can be trivialized by the same cover $M = \bigcup U_\alpha$. Let

$$\tau_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{R}), \quad \tilde{\tau}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{R}),$$

denote the transition functions.

(a) Show that E is isomorphic to $\tilde{E}$ if and only if there exists local functions $h_\alpha : U_\alpha \rightarrow GL(k, \mathbb{R})$ such that

$$\tilde{\tau}_{\alpha\beta} = h_\alpha \tau_{\alpha\beta} h_\beta^{-1},$$

on $U_\alpha \cap U_\beta$.

(b) Show that a line bundle $L \rightarrow M$ is isomorphic to the trivial bundle $M \times \mathbb{R}$ if and only if it admits a nowhere vanishing section.

Solution. We start with part (a). First, we suppose a collection $\{h_\alpha\}$ exists and construct a bundle isomorphism $H \in \Gamma(\text{Hom}(E, \tilde{E}))$. We can define this isomorphism by the following local expressions over the induced trivializations of $\text{Hom}(E, \tilde{E})$:

$$H|_{U_\alpha} := (x, h_\alpha(x)).$$

Since

$$h_\alpha = \tilde{\tau}_{\alpha\beta} h_\beta \tau_{\alpha\beta}^{-1},$$

the local sections h_α glue to a global section of $E^* \otimes \tilde{E} = \text{Hom}(E, \tilde{E})$. Since $h_\alpha \in GL(k, \mathbb{R})$, we know that

$$H(p) : E_p \rightarrow \tilde{E}_p$$

is invertible for all $p \in M$, hence $H \in \Gamma(\text{Hom}(E, \tilde{E}))$ is an isomorphism and $E \cong \tilde{E}$.

For the reverse direction, suppose $E \cong \tilde{E}$ via an isomorphism $H \in \Gamma(\text{Hom}(E, \tilde{E}))$. Writing out this isomorphism in local trivializations, we see

$$H|_{U_\alpha} = (x, h_\alpha(x)),$$

and the $h_\alpha : U_\alpha \rightarrow GL(k, \mathbb{R})$ satisfy $h_\alpha = \tilde{\tau}_{\alpha\beta} h_\beta \tau_{\alpha\beta}^{-1}$ on overlaps.

Next, for part (b), let $L \rightarrow M$ be a line bundle. A nowhere vanishing section $s \in \Gamma(L)$ is equivalent to the data

$$s_\alpha : U_\alpha \rightarrow \mathbb{R}^*, \quad s_\alpha = \tau_{\alpha\beta} s_\beta.$$

Let $h_\alpha = s_\alpha^{-1}$. Then a nowhere vanishing section $s \in \Gamma(L)$ is equivalent to a collection of local functions $h_\alpha : U_\alpha \rightarrow GL(1, \mathbb{R})$ such that

$$1 = h_\alpha \tau_{\alpha\beta} h_\beta^{-1}. \quad (1)$$

By part (a), the relations (1) are equivalent to L being isomorphic to a bundle with transition functions $\tilde{\tau} \equiv 1$, namely the trivial bundle. □

Problem 4. In this problem, we will show that the direct sum of two copies of the Möbius bundle is isomorphic to a trivial bundle. To do this, you may use the cover $S^1 = U_1 \cup U_2$ with

$$U_1 = \{e^{i\theta} : \theta \in (0, 2\pi)\}, \quad U_2 = \{e^{i\tilde{\theta}} : \tilde{\theta} \in (-\pi, \pi)\}$$

with change of coordinates

$$\tilde{\theta} = \begin{cases} \theta, & \text{if } 0 < \theta < \pi \\ \theta - 2\pi, & \text{if } \pi < \theta < 2\pi. \end{cases}$$

As we know from the previous problem, the Möbius bundle $E \rightarrow S^1$ is defined by the data $(U_1 \cap U_2, \tau_{12})$ with

$$\tau_{12} = \begin{cases} 1, & \text{if } 0 < \theta < \pi \\ -1, & \text{if } \pi < \theta < 2\pi. \end{cases}$$

To show $E \oplus E \rightarrow S^1$ is the trivial bundle, find

$$h_\alpha : U_\alpha \rightarrow GL(2, \mathbb{R})$$

such that on $U_1 \cap U_2$ there holds

$$I_{2 \times 2} = h_1 \begin{bmatrix} \tau_{12} & 0 \\ 0 & \tau_{12} \end{bmatrix} h_2^{-1}.$$

Hint: the h_α will be of the form

$$\begin{bmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{bmatrix},$$

where $\psi(\theta)$ on U_1 and $\psi(\tilde{\theta})$ on U_2 .

Solution. Recall that $U_1 \cap U_2$ has two components: an upper circle $\{0 < \theta < \pi\}$ and a lower circle $\{\pi < \theta < 2\pi\}$. We are looking for matrices h_1 and h_2 such that $h_1 = h_2$ on the upper circle and $h_1 = -h_2$ on the lower circle. The following matrices have the required properties. Let

$$h_1 = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}, \quad 0 < \theta < 2\pi,$$

over U_1 , and

$$h_2 = \begin{bmatrix} \cos(\tilde{\theta}/2) & -\sin(\tilde{\theta}/2) \\ \sin(\tilde{\theta}/2) & \cos(\tilde{\theta}/2) \end{bmatrix}, \quad -\pi < \tilde{\theta} < \pi,$$

over U_2 . These matrices solve

$$I_{2 \times 2} = h_1 \tau_{12} h_2^{-1}. \quad (2)$$

Over the upper circle, $\tilde{\theta} = \theta$ and (2) is immediate. Over the lower circle, $\tilde{\theta} = \theta - 2\pi$ and so

$$h_2 = \begin{bmatrix} \cos((\theta - 2\pi)/2) & -\sin((\theta - 2\pi)/2) \\ \sin((\theta - 2\pi)/2) & \cos((\theta - 2\pi)/2) \end{bmatrix} = - \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix},$$

which means $h_2^{-1} = -h_1^{-1}$. This implies (2) since $\tau_{12} = -1$ over the lower circle, and so the pair of minus signs become $+1$. □

Problem 5. Let $\pi : E \rightarrow M$ be a vector bundle and let $s : M \rightarrow E$ be a section. Show that

$$S = \{s(p) : p \in M\}$$

is a submanifold of E which is diffeomorphic to M .

Solution. Denote the dimension of M by n and the rank of E by k . Let x^i be coordinates on M over $U_\alpha \subseteq M$. In a local trivialization $\varphi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^k$, the section appears as

$$s|_{U_\alpha} := \varphi_\alpha \circ s = (x^1, \dots, x^n, s_\alpha^1(x), \dots, s_\alpha^k(x)),$$

where $s_\alpha : U_\alpha \rightarrow \mathbb{R}^k$ is a smooth function. Denote coordinates in the trivialization φ_α by

$$(x^1, \dots, x^n, v^1, \dots, v^k).$$

We can define slice coordinates on $\pi^{-1}(U_\alpha)$ by

$$y^i = s_\alpha^i(x) - v^i.$$

In other words, we have equipped $\pi^{-1}(U_\alpha)$ with new coordinates

$$(x^1, \dots, x^n, y^1, \dots, y^k)$$

such that

$$\pi^{-1}(U_\alpha) \cap S = \{(x, y) \in \pi^{-1}(U_\alpha) : y^1 = \dots = y^k = 0\}$$

and so S is a submanifold. For the diffeomorphism, consider

$$F : M \rightarrow S, \quad F(p) = s(p).$$

This is bijective with inverse $F^{-1} = \pi|_S$. We must check that F is smooth with smooth inverse. In the coordinates constructed above, namely x^i on M and (x^i, y^i) on E , the map takes the local form

$$F(x) \stackrel{\text{loc}}{=} (x, 0),$$

as a map into E . As a map using the coordinates x on $S = \{(x, 0)\} \subseteq E$, the map F is simply $x \mapsto x$. Therefore F is smooth and locally a diffeomorphism. Therefore F is a diffeomorphism. $\square$