

Problem Set 5: Differential Forms

Problem 1. Here we practice manipulations of differential forms.

(a) Let $\alpha = xdx + ydy$ and $\beta = ydx + xdy$ be 1-forms on $\mathbb{R}^2 \setminus \{(x, y) : x = \pm y\}$. Find the frame of vector fields $\{X, Y\}$ such that $\{\alpha, \beta\}$ is the dual frame of $\{X, Y\}$.

(b) Let $\alpha \in \Omega^3(\mathbb{R}^4)$ be

$$\alpha = x \, dy \wedge dz \wedge dw.$$

Compute $\alpha_p(X, Y, Z)$ at $p = (2, 2, 0, -1)$ where

$$X = \frac{\partial}{\partial x} + \frac{\partial}{\partial w}, \quad Y = \frac{\partial}{\partial y} + \frac{\partial}{\partial w}, \quad Z = \frac{\partial}{\partial z} + \frac{\partial}{\partial y},$$

are vector fields on $\mathbb{R}^4$.

(c) Let $h(x, y) = x^2y$ and $g(x, y) = \sin(xy)$ be two functions in $\mathbb{R}^2$. Compute $dh \wedge dg$, which is a 2-form in $\mathbb{R}^2$.

(d) Let $\alpha = \sin x \, dx + y \, dy$ and

$$\beta = (x^2 + yz) \, dx + (xy + z^2) \, dy + (y^2 - xz) \, dz.$$

Compute $d\alpha$, $d\beta$ and $d(\alpha \wedge \beta)$.

(e) Let $X = x\partial_x + y\partial_y + z\partial_z$ and $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$. Compute $X \lrcorner \omega$.

Problem 2. Let $\alpha \in \Omega^1(\mathbb{R}^n)$ be nowhere zero. Let $\chi \in \Omega^k(\mathbb{R}^n)$. Show that $\chi = \alpha \wedge \beta$ for some $\beta \in \Omega^{k-1}(\mathbb{R}^n)$ if and only if $\alpha \wedge \chi = 0$.

Problem 3. Let $\alpha \in \Omega^1(\mathbb{R}^2)$ be a smooth 1-form defined on all of $\mathbb{R}^2$. Suppose that

$$\alpha = f(\theta) \, d\theta, \quad \text{on } \mathbb{R}^2 \setminus \{0\},$$

when written in polar coordinates $x = r \cos \theta$, $y = r \sin \theta$. Show that $\alpha \equiv 0$.

Problem 4. Find the flow of $V = x\partial_y - y\partial_x$ on $\mathbb{R}^2$, and use it to compute the Lie derivative $L_V(x\,dy)$.

Problem 5. Consider the chart $U = S^2 \cap \{z > 0\}$ on $S^2 = \{x^2 + y^2 + z^2 = 1\}$ with coordinates

$$\varphi(x, y, \sqrt{1 - x^2 - y^2}) = (x, y).$$

Let $\iota : S^2 \rightarrow \mathbb{R}^3$ be inclusion.

- (a) Compute the local coordinate expression for ι^*dx , ι^*dy and ι^*dz over (U, φ) .
 - (b) Compute the local coordinate expression for $\iota^*(x\,dx + y\,dy + z\,dz)$ over (U, φ) by using part (a). Is there another way to anticipate this result?
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Problem 6. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a smooth function such that

$$\frac{\partial f}{\partial z}(p) \neq 0$$

at all points $p \in S$, where $S = f^{-1}(c)$.

- (a) State the theorem from which you can deduce that S is a manifold.
 - (b) Let $\omega \in \Omega^2(\mathbb{R}^3)$ be $\omega = dx \wedge dy$. Let $\iota : S \rightarrow \mathbb{R}^3$. Show that $\iota^*\omega \in \Omega^2(S)$ is nowhere vanishing.
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Problem 7. Let $S^2 = \{x^2 + y^2 + z^2 = 1\}$ and $\iota : S^2 \rightarrow \mathbb{R}^3$ be inclusion. Consider

$$\begin{aligned}\tilde{\omega} &= x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy \in \Omega^2(\mathbb{R}^3) \\ \tilde{\eta} &= -y\,dx + x\,dy \in \Omega^1(\mathbb{R}^3).\end{aligned}$$

Define $\omega = \iota^*\tilde{\omega} \in \Omega^2(S^2)$ and $\eta = \iota^*\tilde{\eta} \in \Omega^1(S^2)$.

- (a) Find $\lambda(z) \in C^\infty(S^2)$ such that $d\eta = \lambda\omega$.
- (b) Find $\phi(z) \in C^\infty(S^2)$ such that $X \lrcorner \omega \wedge Y \lrcorner \omega = \phi\omega$, where

$$\tilde{X} = -zx\frac{\partial}{\partial x} - zy\frac{\partial}{\partial y} + (x^2 + y^2)\frac{\partial}{\partial z}, \quad \tilde{Y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

are vector fields on $\mathbb{R}^3$ which when restricted to the sphere define vector fields X and Y on S^2 .

Problem 8. Let $\{i\sigma_1, i\sigma_2, i\sigma_3\}$ denote the following basis of $\mathfrak{su}(2)$:

$$i\sigma_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad i\sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad i\sigma_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Extend this basis to a frame of left-invariant vector fields, denoted $\{e_1, e_2, e_3\}$, on all of $SU(2)$. In explicit detail, this means

$$e_k|_g = (dL_g)|_I(i\sigma_k),$$

where $L_g : SU(2) \rightarrow SU(2)$ is left multiplication $h \mapsto gh$. Let $\{e^1, e^2, e^3\}$ be the dual basis of 1-forms on $SU(2)$. Compute de^1 . You can use the commutator identities

$$[i\sigma_1, i\sigma_2] = -2i\sigma_3, \quad [i\sigma_2, i\sigma_3] = -2i\sigma_1, \quad [i\sigma_3, i\sigma_1] = -2i\sigma_2.$$

Problem 9. Let X be a vector field on a compact manifold M with flow $\Theta_t : M \rightarrow M$. Let $\alpha \in \Omega^k(M)$. Show that if $L_X\alpha = 0$ then $\Theta_t^*\alpha = \alpha$.

Problem 10. Let $\omega \in \Omega^2(M)$, written in local components as

$$\omega \stackrel{\text{loc}}{=} \frac{1}{2} \omega_{ij} dx^i \wedge dx^j.$$

If $d\omega = 0$ and $\omega_{ij}(x)$ is an invertible skew-symmetric matrix at every point x , then ω is said to be a symplectic form.

(a) Show this definition is well-defined. This amounts to checking that if x and $\tilde{x}$ are two sets of local coordinates over an open set $U \subseteq M$, then if ω_{ij} is invertible in coordinates x then $\tilde{\omega}_{\alpha\beta}$ is invertible in coordinates $\tilde{x}$.

(b) Show that $\omega \in \Omega^2(M)$ with $d\omega = 0$ is a symplectic form if and only if the following condition holds: for every $p \in M$ and $V \in T_pM$, then the condition

$$\omega_p(V, W) = 0 \quad \text{for all } W \in T_pM$$

implies that $V = 0$.

(c) Let (M, ω) be a symplectic manifold. Given $f \in C^\infty(M)$, the Hamiltonian vector field of f is defined by

$$V = V^i \partial_i, \quad V^i = \omega^{ik} \partial_k f, \tag{1}$$

where ω^{ik} are the matrix entries of the inverse of ω_{ij} : this means $\omega^{ik} \omega_{kj} = \delta^i_j$. Show that V satisfies

$$V \lrcorner \omega = -df.$$

Conclude that the local vector fields defined in (1) agree on overlaps: $V^i \partial_{x^i} = \tilde{V}^\alpha \partial_{\tilde{x}^\alpha}$.

(d) Let

$$\omega = \sum_{i=1}^n dx^i \wedge dy^i,$$

on $\mathbb{R}^{2n}$. Let V be the Hamiltonian vector field of a function f . Compute the components of V in terms of f .

(e) Let V be a Hamiltonian vector field on a symplectic manifold (M, ω) . Show that $L_V \omega = 0$ and hence the flow of V preserves ω .

Problem 11. Let $\widetilde{M}$ be a compact manifold with a diffeomorphism $\iota : \widetilde{M} \rightarrow \widetilde{M}$ satisfying the involution condition

$$\iota^2 = \text{id}$$

and the fixed-point free condition

$$\iota(p) \neq p \quad \text{for all } p \in \widetilde{M}.$$

In this setup, the quotient

$$M = \widetilde{M} / \langle \iota \rangle$$

is also a smooth manifold with local diffeomorphism

$$\pi : \widetilde{M} \rightarrow M, \quad x \mapsto [x].$$

Show that

$$\pi^* : \Omega^k(M) \rightarrow \left\{ \alpha \in \Omega^k(\widetilde{M}) : \iota^* \alpha = \alpha \right\}$$

is a bijection.
