

Problem Set 5: Differential Forms

Problem 1. Here we practice manipulations of differential forms.

(a) Let $\alpha = xdx + ydy$ and $\beta = ydx + xdy$ be 1-forms on $\mathbb{R}^2 \setminus \{(x, y) : x = \pm y\}$. Find the frame of vector fields $\{X, Y\}$ such that $\{\alpha, \beta\}$ is the dual frame of $\{X, Y\}$.

(b) Let $\alpha \in \Omega^3(\mathbb{R}^4)$ be

$$\alpha = x \, dy \wedge dz \wedge dw.$$

Compute $\alpha_p(X, Y, Z)$ at $p = (2, 2, 0, -1)$ where

$$X = \frac{\partial}{\partial x} + \frac{\partial}{\partial w}, \quad Y = \frac{\partial}{\partial y} + \frac{\partial}{\partial w}, \quad Z = \frac{\partial}{\partial z} + \frac{\partial}{\partial y},$$

are vector fields on $\mathbb{R}^4$.

(c) Let $h(x, y) = x^2y$ and $g(x, y) = \sin(xy)$ be two functions in $\mathbb{R}^2$. Compute $dh \wedge dg$, which is a 2-form in $\mathbb{R}^2$.

(d) Let $\alpha = \sin x \, dx + y \, dy$ and

$$\beta = (x^2 + yz) \, dx + (xy + z^2) \, dy + (y^2 - xz) \, dz.$$

Compute $d\alpha$, $d\beta$ and $d(\alpha \wedge \beta)$.

(e) Let $X = x\partial_x + y\partial_y + z\partial_z$ and $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$. Compute $X \lrcorner \omega$.

Solution. For (a), write

$$X = X^1\partial_x + X^2\partial_y, \quad Y = Y^1\partial_x + Y^2\partial_y.$$

The equations $\alpha(X) = 1$, $\alpha(Y) = 0$ are

$$1 = xX^1 + yX^2, \quad 0 = yX^1 + xX^2$$

which imply

$$X = \frac{x}{x^2 - y^2} \frac{\partial}{\partial x} - \frac{y}{x^2 - y^2} \frac{\partial}{\partial y}.$$

The equations $\alpha(Y) = 0$ and $\beta(Y) = 1$ are

$$0 = xY^1 + yY^2, \quad 1 = yY^1 + xY^2$$

which imply

$$Y = -\frac{y}{x^2 - y^2} \frac{\partial}{\partial x} + \frac{x}{x^2 - y^2} \frac{\partial}{\partial y}.$$

For (b), we expand

$$\alpha_p(X, Y, Z) = 2dy \wedge dz \wedge dw \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial w}, \frac{\partial}{\partial y} + \frac{\partial}{\partial w}, \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right).$$

We drop the ∂_x in the first slot. We use linearity on the second slot.

$$\begin{aligned} \alpha_p(X, Y, Z) &= 2dy \wedge dz \wedge dw \left(\frac{\partial}{\partial w}, \frac{\partial}{\partial y}, \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \\ &\quad + 2dy \wedge dz \wedge dw \left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \end{aligned}$$

The second term is zero due to repeating ∂_w . We use linearity on the third slot and the drop term with repeating ∂_y .

$$\alpha_p(X, Y, Z) = 2dy \wedge dz \wedge dw \left(\frac{\partial}{\partial w}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right),$$

Skew-symmetry in the order of inputs gives

$$\alpha_p(X, Y, Z) = 2dy \wedge dz \wedge dw \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial w} \right) = 2.$$

For (c), we compute

$$dh = 2xydx + x^2dy, \quad dg = y \cos(xy)dx + x \cos(xy)dy$$

and so

$$dh \wedge dg = (2xydx + x^2dy) \wedge (y \cos(xy)dx + x \cos(xy)dy)$$

which becomes

$$dh \wedge dg = 2x^2y \cos(xy)dx \wedge dy + x^2y \cos(xy)dy \wedge dx$$

and so

$$dh \wedge dg = \left[x^2y \cos(xy) \right] dx \wedge dy.$$

For (d), we start with $\alpha = \sin x dx + y dy$ which gives

$$d\alpha = 0,$$

and

$$\begin{aligned} d\beta &= \partial_y(yz)dy \wedge dx + \partial_z(yz)dz \wedge dx + \partial_x(xy + z^2)dx \wedge dy + \partial_z(xy + z^2)dz \wedge dy \\ &\quad + \partial_x(y^2 - xz)dx \wedge dz + \partial_y(y^2 - xz)dy \wedge dz. \end{aligned}$$

Therefore

$$d\beta = zdy \wedge dx + ydz \wedge dx + ydx \wedge dy + 2zdz \wedge dy - zdx \wedge dz + 2ydy \wedge dz,$$

which simplifies to

$$d\beta = (y - z) dx \wedge dy + (-y - z) dx \wedge dz + (-2z + 2y) dy \wedge dz.$$

Next, since $\deg \alpha = 1$ and $d\alpha = 0$,

$$d(\alpha \wedge \beta) = -\alpha \wedge d\beta$$

which becomes

$$d(\alpha \wedge \beta) = \left(-(\sin x)(-2z + 2y) - y(y + z) \right) dx \wedge dy \wedge dz.$$

For (e), the interior product is

$$X \lrcorner \omega = x X \lrcorner (dy \wedge dz) + y X \lrcorner (dz \wedge dx) + z X \lrcorner (dx \wedge dy),$$

which by Leibniz rule becomes

$$\begin{aligned} X \lrcorner \omega &= x(X \lrcorner dy) \wedge dz - x dy \wedge (X \lrcorner dz) + y(X \lrcorner dz) \wedge dx - y dz \wedge (X \lrcorner dx) \\ &\quad + z(X \lrcorner dx) \wedge dy - z dx \wedge (X \lrcorner dy), \end{aligned}$$

with $X = x\partial_x + y\partial_y + z\partial_z$. This evaluates to

$$X \lrcorner \omega = xydz - xzdy + yzdx - x y dz + x z dy - y z dx,$$

and all terms cancel.

$$X \lrcorner \omega = 0.$$

□

Problem 2. Let $\alpha \in \Omega^1(\mathbb{R}^n)$ be nowhere zero. Let $\chi \in \Omega^k(\mathbb{R}^n)$. Show that $\chi = \alpha \wedge \beta$ for some $\beta \in \Omega^{k-1}(\mathbb{R}^n)$ if and only if $\alpha \wedge \chi = 0$.

Solution. Suppose χ can be written as $\chi = \alpha \wedge \beta$. Then

$$\alpha \wedge \chi = \alpha \wedge \alpha \wedge \beta = 0$$

since $\alpha \wedge \alpha = 0$ for any 1-form α . On the other hand, suppose $\alpha \wedge \chi = 0$. Let

$$\alpha = \alpha_i dx^i$$

and define the vector field

$$V = \sum_i \frac{\alpha_i}{\sum_k \alpha_k^2} \frac{\partial}{\partial x^i}$$

so that $\alpha(V) = 1$. Then the interior product of $\alpha \wedge \chi$ gives

$$0 = V \lrcorner (\alpha \wedge \chi) = (V \lrcorner \alpha) \wedge \chi - \alpha \wedge (V \lrcorner \chi)$$

which gives

$$\alpha \wedge (V \lrcorner \chi) = \chi.$$

Therefore $\beta = (V \lrcorner \chi)$.

□

Problem 3. Let $\alpha \in \Omega^1(\mathbb{R}^2)$ be a smooth 1-form defined on all of $\mathbb{R}^2$. Suppose that

$$\alpha = f(\theta) d\theta, \quad \text{on } \mathbb{R}^2 \setminus \{0\},$$

when written in polar coordinates $x = r \cos \theta$, $y = r \sin \theta$. Show that $\alpha \equiv 0$.

Solution. The change of coordinates formula is

$$d\theta = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy.$$

We must calculate $\partial_x \theta$ and $\partial_y \theta$. Differentiating $x = r \cos \theta$, $y = r \sin \theta$ with ∂_x gives

$$1 = (\partial_x r) \cos \theta - (r \sin \theta) \partial_x \theta, \quad 0 = (\partial_x r) \sin \theta + (r \cos \theta) \partial_x \theta,$$

and multiplying the first equation by $\sin \theta$, the second by $\cos \theta$, and subtracting equations gives

$$\sin \theta = -r \partial_x \theta \implies \partial_x \theta = -\frac{\sin \theta}{r} \implies \partial_x \theta = -\frac{y}{x^2 + y^2}$$

A similar calculation gives

$$\partial_y \theta = \frac{x}{x^2 + y^2}.$$

Therefore

$$d\theta = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy,$$

and

$$\alpha = -\frac{yf(\theta(x, y))}{x^2 + y^2} dx + \frac{xf(\theta(x, y))}{x^2 + y^2} dy, \quad \text{over } \mathbb{R}^2 \setminus \{0\}.$$

Since α is smooth over all of $\mathbb{R}^2$, then

$$a_1(x, y) = -\frac{yf(\theta(x, y))}{x^2 + y^2}, \quad a_2(x, y) = \frac{xf(\theta(x, y))}{x^2 + y^2}$$

are smooth functions on $\mathbb{R}^2$. We probe them by the ray $(x(t), y(t)) = (t \cos \theta_0, t \sin \theta_0)$ for a fixed angle θ_0 as $t \rightarrow 0$.

$$a_1(0, 0) = \lim_{t \rightarrow 0} \frac{(\cos \theta_0)f(\theta_0)}{t}, \quad a_2(0, 0) = \lim_{t \rightarrow 0} \frac{(\sin \theta_0)f(\theta_0)}{t}.$$

We cannot have both $\cos \theta_0 = 0$ and $\sin \theta_0 = 0$, so we must have $f(\theta_0) = 0$. This holds for any θ_0 , hence $f \equiv 0$.

□

Problem 4. Find the flow of $V = x\partial_y - y\partial_x$ on $\mathbb{R}^2$, and use it to compute the Lie derivative $L_V(x dy)$.

Solution. We must solve $\frac{d}{dt}(x(t), y(t)) = V_{x(t), y(t)}$, which is the ODE

$$\dot{x} = -y, \quad \dot{y} = x$$

with initial condition (x_0, y_0) . Taking a derivative of $\dot{x} = -y$ and substituting $\dot{y}$ gives

$$\ddot{x} = -x.$$

This has solution

$$x(t) = c \sin t + x_0 \cos t.$$

Substituting this $x(t)$ into $\dot{x} = -y$ gives

$$y(t) = -(c \cos t - x_0 \sin t) \implies c = -y_0.$$

Therefore

$$x(t) = x_0 \cos t - y_0 \sin t, \quad y(t) = x_0 \sin t + y_0 \cos t.$$

Written in the notation $\Theta_t(x, y) = (x(t), y(t))$, this is

$$\Theta_t(x, y) = (x \cos t - y \sin t, x \sin t + y \cos t).$$

The Lie derivative is

$$L_V(xdy) = \frac{d}{dt} \Big|_{t=0} \Theta_t^*(xdy).$$

This is

$$L_V(xdy) = \frac{d}{dt} \Big|_{t=0} \left((x \cos t - y \sin t)(\sin t dx + \cos t dy) \right)$$

which expands to

$$L_V(xdy) = (-x \sin t - y \cos t)(\sin t dx + \cos t dy) + (x \cos t - y \sin t)(\cos t dx - \sin t dy) \Big|_{t=0}$$

and simplifies to

$$L_V(xdy) = xdx - ydy.$$

□

Problem 5. Consider the chart $U = S^2 \cap \{z > 0\}$ on $S^2 = \{x^2 + y^2 + z^2 = 1\}$ with coordinates

$$\varphi(x, y, \sqrt{1 - x^2 - y^2}) = (x, y).$$

Let $\iota : S^2 \rightarrow \mathbb{R}^3$ be inclusion.

- (a) Compute the local coordinate expression for ι^*dx , ι^*dy and ι^*dz over (U, φ) .
- (b) Compute the local coordinate expression for $\iota^*(xdx + ydy + zdz)$ over (U, φ) by using part (a). Is there another way to anticipate this result?

Solution. We compute

$$\iota^* dx = dx, \quad \iota^* dy = dy,$$

and

$$\iota^* dz = d\sqrt{1-x^2-y^2} = -\frac{x}{\sqrt{1-x^2-y^2}} dx - \frac{y}{\sqrt{1-x^2-y^2}} dy.$$

Therefore

$$\iota^*(x dx + y dy + z dz) = x dx + y dy + \sqrt{1-x^2-y^2} \left[-\frac{x}{\sqrt{1-x^2-y^2}} dx - \frac{y}{\sqrt{1-x^2-y^2}} dy \right]$$

and so

$$\iota^*(x dx + y dy + z dz) = 0.$$

This is because $r^2 = x^2 + y^2 + z^2$ satisfies

$$dr^2 = 2(x dx + y dy + z dz)$$

and

$$\iota^* dr^2 = d\iota^* r^2 = d(1) = 0$$

since $r = 1$ on S^2 . □

Problem 6. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a smooth function such that

$$\frac{\partial f}{\partial z}(p) \neq 0$$

at all points $p \in S$, where $S = f^{-1}(c)$.

(a) State the theorem from which you can deduce that S is a manifold.

(b) Let $\omega \in \Omega^2(\mathbb{R}^3)$ be $\omega = dx \wedge dy$. Let $\iota : S \rightarrow \mathbb{R}^3$. Show that $\iota^* \omega \in \Omega^2(S)$ is nowhere vanishing.

Solution. For (a), one can prove this either using the regular level set theorem or the implicit function theorem. The regular level set theorem says that if $f : M \rightarrow N$ is a smooth map and $c \in N$ is a regular value, then $f^{-1}(c)$ is a submanifold. In this case c is a regular value because for $p \in f^{-1}(c)$ then

$$df|_p = (\partial_x f(p) \quad \partial_y f(p) \quad \partial_z f(p))$$

has maximal rank since $\partial_z f(p) \neq 0$.

Using the implicit function theorem, there exists a neighborhood $U \subset \mathbb{R}^3$ of p and an open set $V \subset \mathbb{R}^2$ and a smooth function $\psi : V \rightarrow \mathbb{R}$ such that

$$U \cap S = \{(x, y, \psi(x, y)) : (x, y) \in V\}.$$

We may equip S with coordinates $\varphi : U \cap S \rightarrow V \subset \mathbb{R}^2$

$$\varphi(x, y, \psi(x, y)) = (x, y).$$

In these coordinates $\iota(x, y) = (x, y, \psi(x, y))$. Therefore

$$\iota^*\omega|_V = dx \wedge dy.$$

In these coordinates it is clear that $\iota^*\omega$ is a nowhere vanishing top form. □

Problem 7. Let $S^2 = \{x^2 + y^2 + z^2 = 1\}$ and $\iota : S^2 \rightarrow \mathbb{R}^3$ be inclusion. Consider

$$\begin{aligned}\tilde{\omega} &= xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \in \Omega^2(\mathbb{R}^3) \\ \tilde{\eta} &= -ydx + xdy \in \Omega^1(\mathbb{R}^3).\end{aligned}$$

Define $\omega = \iota^*\tilde{\omega} \in \Omega^2(S^2)$ and $\eta = \iota^*\tilde{\eta} \in \Omega^1(S^2)$.

(a) Find $\lambda(z) \in C^\infty(S^2)$ such that $d\eta = \lambda\omega$.

(b) Find $\phi(z) \in C^\infty(S^2)$ such that $X \lrcorner \omega \wedge Y \lrcorner \omega = \phi\omega$, where

$$\tilde{X} = -zx \frac{\partial}{\partial x} - zy \frac{\partial}{\partial y} + (x^2 + y^2) \frac{\partial}{\partial z}, \quad \tilde{Y} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y},$$

are vector fields on $\mathbb{R}^3$ which when restricted to the sphere define vector fields X and Y on S^2 .

Solution. We start by computing $\omega = \iota^*\tilde{\omega}$ on the upper hemisphere $\{z > 0\}$ using coordinates $(x, y, \sqrt{1 - x^2 - y^2})$. We computed ι^*dz in an earlier problem in this set, and our coordinates satisfy $\iota^*dx = dx$ and $\iota^*dy = dy$. Therefore

$$\begin{aligned}\omega &= xdy \wedge \left[-\frac{x}{\sqrt{1 - x^2 - y^2}} dx - \frac{y}{\sqrt{1 - x^2 - y^2}} dy \right] \\ &\quad + y \left[-\frac{x}{\sqrt{1 - x^2 - y^2}} dx - \frac{y}{\sqrt{1 - x^2 - y^2}} dy \right] \wedge dx \\ &\quad + \left[\sqrt{1 - x^2 - y^2} \right] dx \wedge dy.\end{aligned}$$

This simplifies to

$$\omega = \frac{1}{\sqrt{1 - x^2 - y^2}} dx \wedge dy = \frac{1}{z} dx \wedge dy \quad \text{on } \{z > 0\}.$$

On the other hand, in these coordinates

$$\eta = -ydx + xdy, \quad d\eta = 2dx \wedge dy.$$

So

$$\lambda = 2\sqrt{1 - x^2 - y^2} = 2z, \quad \text{on } \{z > 0\}.$$

This is only on the upper half of S^2 . On the lower half $\{z < 0\}$, we use the parametrization $(x, y, -\sqrt{1 - x^2 - y^2})$ so that

$$\omega = -\frac{1}{\sqrt{1 - x^2 - y^2}} dx \wedge dy.$$

On the lower half, η is the same, and so

$$\lambda = -2\sqrt{1 - x^2 - y^2} = 2z \quad \text{on } \{z < 0\}.$$

The general formula is thus

$$\lambda = 2z.$$

We derived this formula on $\{z \neq 0\} \cap S^2$, but in fact $\lambda = 2z$ on all of S^2 since both sides are continuous functions.

For part (b), the vector fields $\tilde{X}$ and $\tilde{Y}$ are tangential to the sphere since

$$\tilde{X} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0, \quad \tilde{Y} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

We write these S^2 -vector fields in (x, y) coordinates over $\{z > 0\}$.

$$X = -zx\partial_x - zy\partial_y, \quad Y = -y\partial_x + x\partial_y.$$

If this step is unclear, see below for a full explanation. Next, as before, ω in (x, y) coordinates on $\{z > 0\}$ is

$$\omega = \frac{1}{z} dx \wedge dy.$$

So

$$X \lrcorner \omega = -x dy + y dx, \quad Y \lrcorner \omega = -\frac{y}{z} dy - \frac{x}{z} dx,$$

and

$$(X \lrcorner \omega) \wedge (Y \lrcorner \omega) = -\frac{x^2 + y^2}{z} dx \wedge dy.$$

Therefore we can read-off

$$\phi = -x^2 - y^2 = z^2 - 1,$$

over $\{z > 0\}$. The same calculation works on $\{z < 0\}$. By continuity, we extend the identity across the equator and deduce

$$\phi = z^2 - 1 \quad \text{on } S^2.$$

Let us now understand why in coordinates (U, φ) with $\varphi^{-1}(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$, then

$$X|_U = -zx\partial_x - zy\partial_y$$

is the coordinate representation of the tangent vector

$$\tilde{X} = \begin{bmatrix} -zx \\ -zy \\ x^2 + y^2 \end{bmatrix} \in \mathbb{R}^3.$$

Here is the general theory. If

$$\tilde{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

is a tangent vector in $\mathbb{R}^3$ coming from some path $c(t)$ on the sphere $S^2 \subseteq \mathbb{R}^3$ (say on the upper-hemisphere), then

$$\tilde{X} = \frac{d}{dt} \Big|_{t=0} c(t) = \frac{d}{dt} \Big|_{t=0} \begin{bmatrix} x(t) \\ y(t) \\ \sqrt{1 - x(t)^2 - y(t)^2} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ -\frac{x\dot{x} + y\dot{y}}{z} \end{bmatrix}.$$

In local coordinates (U, φ) the path $c(t)$ appears as $(x(t), y(t)) = \varphi \circ c(t)$, or in our usual notation

$$c(t) \stackrel{\text{loc}}{=} (x(t), y(t)).$$

The vector field in coordinates is the derivative of the path in coordinates. So

$$X|_U = \frac{d}{dt} \Big|_{t=0} (x(t), y(t)) = \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y}.$$

For the given example

$$\tilde{X} = \begin{bmatrix} -zx \\ -zy \\ x^2 + y^2 \end{bmatrix} \in \mathbb{R}^3,$$

we have $\dot{x} = -zx$ and $\dot{y} = -zy$, and $c(t) = \frac{p+t\tilde{X}_p}{|p+t\tilde{X}_p|}$ at $p = (x, y, z) \in S^2$. □

Problem 8. Let $\{i\sigma_1, i\sigma_2, i\sigma_3\}$ denote the following basis of $\mathfrak{su}(2)$:

$$i\sigma_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad i\sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad i\sigma_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Extend this basis to a frame of left-invariant vector fields, denoted $\{e_1, e_2, e_3\}$, on all of $SU(2)$. In explicit detail, this means

$$e_k|_g = (dL_g)|_I(i\sigma_k),$$

where $L_g : SU(2) \rightarrow SU(2)$ is left multiplication $h \mapsto gh$. Let $\{e^1, e^2, e^3\}$ be the dual basis of 1-forms on $SU(2)$. Compute de^1 . You can use the commutator identities

$$[i\sigma_1, i\sigma_2] = -2i\sigma_3, \quad [i\sigma_2, i\sigma_3] = -2i\sigma_1, \quad [i\sigma_3, i\sigma_1] = -2i\sigma_2.$$

Solution. We use the coordinate-free formula for the exterior derivative.

$$(de^1)(X, Y) = X(e^1(Y)) - Y(e^1(X)) - e^1([X, Y]).$$

Since $e^k(e_j) = \delta^k_j$, we have

$$(de^1)(e_j, e_k) = -e^1([e_j, e_k]).$$

So we must compute some commutators $[e_j, e_k]$. Note

$$[e_j, e_k] = [(L_g)_*(i\sigma_j), (L_g)_*(i\sigma_k)] = (L_g)_*[i\sigma_j, i\sigma_k],$$

so if

$$[i\sigma_j, i\sigma_k] = c_{jk}^\ell(i\sigma_\ell),$$

for the explicit constants c_{jk}^ℓ given in the problem statement, then

$$[e_j, e_k] = c_{jk}^\ell e_\ell.$$

In other words,

$$[e_1, e_2] = -2e_1, \quad [e_2, e_3] = -2e_1, \quad [e_3, e_1] = -2e_2.$$

Therefore

$$(de^1)(e_1, e_2) = 0, \quad (de^1)(e_1, e_3) = 0, \quad (de^1)(e_2, e_3) = -2.$$

It follows that

$$de^1 = 2e^2 \wedge e^3.$$

□

Problem 9. Let X be a vector field on a compact manifold M with flow $\Theta_t : M \rightarrow M$. Let $\alpha \in \Omega^k(M)$. Show that if $L_X\alpha = 0$ then $\Theta_t^*\alpha = \alpha$.

Solution. The definition of the Lie derivative is

$$\frac{d}{dt}\Theta_t^*\alpha = \Theta_t^*(L_X\alpha).$$

If $L_X\alpha = 0$, then

$$\frac{d}{dt}\Theta_t^*\alpha = 0.$$

Therefore

$$\Theta_t^*\alpha = \Theta_0^*\alpha$$

and since $\Theta_0 = id$, we have $\Theta_t^*\alpha = \alpha$.

□

Problem 10. Let $\omega \in \Omega^2(M)$, written in local components as

$$\omega \stackrel{\text{loc}}{=} \frac{1}{2}\omega_{ij} dx^i \wedge dx^j.$$

If $d\omega = 0$ and $\omega_{ij}(x)$ is an invertible skew-symmetric matrix at every point x , then ω is said to be a symplectic form.

(a) Show this definition is well-defined. This amounts to checking that if x and $\tilde{x}$ are two sets of local coordinates over an open set $U \subseteq M$, then if ω_{ij} is invertible in coordinates x then $\tilde{\omega}_{\alpha\beta}$ is invertible in coordinates $\tilde{x}$.

(b) Show that $\omega \in \Omega^2(M)$ with $d\omega = 0$ is a symplectic form if and only if the following condition holds: for every $p \in M$ and $V \in T_p M$, then the condition

$$\omega_p(V, W) = 0 \quad \text{for all } W \in T_p M$$

implies that $V = 0$.

(c) Let (M, ω) be a symplectic manifold. Given $f \in C^\infty(M)$, the Hamiltonian vector field of f is defined by

$$V = V^i \partial_i, \quad V^i = \omega^{ik} \partial_k f, \quad (1)$$

where ω^{ik} are the matrix entries of the inverse of ω_{ij} : this means $\omega^{ik} \omega_{kj} = \delta^i_j$. Show that V satisfies

$$V \lrcorner \omega = -df.$$

Conclude that the local vector fields defined in (1) agree on overlaps: $V^i \partial_{x^i} = \tilde{V}^\alpha \partial_{\tilde{x}^\alpha}$.

(d) Let

$$\omega = \sum_{i=1}^n dx^i \wedge dy^i,$$

on $\mathbb{R}^{2n}$. Let V be the Hamiltonian vector field of a function f . Compute the components of V in terms of f .

(e) Let V be a Hamiltonian vector field on a symplectic manifold (M, ω) . Show that $L_V \omega = 0$ and hence the flow of V preserves ω .

Solution. For (a), the change of coordinates formula is

$$\tilde{\omega}_{\alpha\beta} = \frac{\partial x^i}{\partial \tilde{x}^\alpha} \omega_{ij} \frac{\partial x^j}{\partial \tilde{x}^\beta}.$$

We write this using matrix multiplication as

$$\tilde{\Omega} = P^T \Omega P,$$

where

$$\Omega = [\omega_{ij}], \quad \tilde{\Omega} = [\tilde{\omega}_{\alpha\beta}], \quad P = \left[\frac{\partial x^i}{\partial \tilde{x}^\alpha} \right].$$

A matrix A is invertible if and only if $\det A \neq 0$. Taking the determinant of both sides gives

$$\det \tilde{\Omega} = (\det P)^2 \det \Omega.$$

Since P is an invertible matrix, we see that

$$\det[\tilde{\omega}_{\alpha\beta}] \neq 0 \iff \det[\omega_{ij}] \neq 0,$$

as required.

For (b), first suppose that $\omega_{ij}(p)$ is an invertible matrix. Suppose $v \in T_p M$ is such that

$$\omega_p(v, \cdot) \equiv 0.$$

Since $\omega_p(v, \partial_k) = 0$, we obtain

$$\omega_{ik}v^i = 0.$$

This is an equation of the form $Av = 0$ where the matrix is $A = [-\omega_{ij}]$. Since A is invertible, it follows that $v = 0$.

Conversely, suppose $\omega_p(v, \cdot) \equiv 0$ implies $v = 0$. If $[\omega_{ij}(p)]$ is not invertible, then it has a kernel. So there exists $v \in T_pM$ with $v \neq 0$ such that

$$\omega_{ik}(p)v^i = 0,$$

for every k . This means $\omega_p(v, \partial_k) = 0$ for every k , hence $\omega_p(v, \cdot) \equiv 0$. This is a contradiction.

For (c), we compute

$$\begin{aligned} (V \lrcorner \omega)(W) &= V^i \omega_{ip} W^p \\ &= \omega^{ik} \partial_k f \omega_{ip} W^p \\ &= -(\partial_k f) W^p \omega^{ki} \omega_{ip} \\ &= -(\partial_k f) W^p \delta_p^k \\ &= -(\partial_p f) W^p. \end{aligned}$$

Since $df(W) = (\partial_p f) W^p$ by definition, we conclude

$$(V \lrcorner \omega)(W) = -df(W) \quad \text{for all } W \in \Gamma(TM),$$

as required. Next, suppose we have two choices of coordinates x^i and $\tilde{x}^\alpha$ and use the local definition to create the local vectors V^i and $\tilde{V}^\alpha$. It follows that

$$\left(V^i \frac{\partial}{\partial x^i} - \tilde{V}^\alpha \frac{\partial}{\partial \tilde{x}^\alpha} \right) \lrcorner \omega = -df + df = 0.$$

By part (b), this implies $V^i \partial_i = \tilde{V}^\alpha \partial_\alpha$. Hence the local definition of V produces a global vector field $V \in \Gamma(TM)$.

For (d), we let

$$V = X^i \frac{\partial}{\partial x^i} + Y^i \frac{\partial}{\partial y^i}$$

and expand

$$\left(X^i \frac{\partial}{\partial x^i} + Y^i \frac{\partial}{\partial y^i} \right) \lrcorner \omega = -df$$

to become

$$\sum_{i=1}^n (X^i dy^i - Y^i dx^i) = -\frac{\partial f}{\partial x^i} dx^i - \frac{\partial f}{\partial y^i} dy^i.$$

We can read-off

$$X^i = -\frac{\partial f}{\partial y^i}, \quad Y^i = \frac{\partial f}{\partial x^i}$$

and

$$V = -\frac{\partial f}{\partial y^i} \frac{\partial}{\partial x^i} + \frac{\partial f}{\partial x^i} \frac{\partial}{\partial y^i}.$$

For (e), we use Cartan's magic formula.

$$L_V\omega = V \lrcorner d\omega + d(V \lrcorner \omega).$$

Since $d\omega = 0$ and $V \lrcorner \omega = -df$, this is

$$L_V\omega = -d^2f = 0.$$

□

Problem 11. Let $\widetilde{M}$ be a compact manifold with a diffeomorphism $\tau : \widetilde{M} \rightarrow \widetilde{M}$ satisfying the involution condition

$$\tau^2 = \text{id}$$

and the fixed-point free condition

$$\tau(p) \neq p \quad \text{for all } p \in \widetilde{M}.$$

In this setup, the quotient

$$M = \widetilde{M}/\langle \tau \rangle$$

is also a smooth manifold with local diffeomorphism

$$\pi : \widetilde{M} \rightarrow M, \quad x \mapsto [x].$$

Show that

$$\pi^* : \Omega^k(M) \rightarrow \left\{ \alpha \in \Omega^k(\widetilde{M}) : \tau^*\alpha = \alpha \right\}$$

is a bijection.

Solution. First, we show that π^* sends $\Omega^k(M)$ to

$$\{\alpha \in \Omega^k(\widetilde{M}) : \tau^*\alpha = \alpha\}.$$

Indeed, if $\beta \in \Omega^k(M)$ then

$$\tau^*(\pi^*\beta) = (\pi \circ \tau)^*\beta = \pi^*\beta$$

as needed. Next, we construct an inverse for π^* . Define

$$\pi_* : \left\{ \alpha \in \Omega^k(\widetilde{M}) : \tau^*\alpha = \alpha \right\} \rightarrow \Omega^k(M)$$

by

$$(\pi_*\alpha)|_p(v_1, \dots, v_k) = \alpha|_{\tilde{p}}(\tilde{v}_1, \dots, \tilde{v}_k), \tag{2}$$

where $\tilde{p} \in \pi^{-1}(p) = \{\tilde{p}, \tau(\tilde{p})\}$ and $\tilde{v}_k \in T_{\tilde{p}}\widetilde{M}$ solves

$$d\pi|_{\tilde{p}}(\tilde{v}_k) = v_k,$$

which is uniquely solvable since $d\pi$ is an isomorphism due to π being a local diffeomorphism. We show this is well-defined. There is one other possible lift of p to $\widetilde{M}$, namely $\tau(\tilde{p})$, with lifted vectors $d\tau(\tilde{v}_k)$. If we used these choices, then we would define

$$(\pi_*\alpha)|_p(v_1, \dots, v_k) = \alpha|_{\tau(\tilde{p})}(d\tau(\tilde{v}_1), \dots, d\tau(\tilde{v}_k)) = (\tau^*\alpha)_{\tilde{p}}(\tilde{v}_1, \dots, \tilde{v}_k)$$

but since $\tau^*\alpha = \alpha$ this agrees with (2). Hence π_* is well-defined. It is direct from the definitions that

$$\pi_*\pi^*\beta = \beta, \quad \pi^*\pi_*\alpha = \alpha.$$

Therefore

$$\pi^* : \Omega^k(M) \rightarrow \left\{ \alpha \in \Omega^k(\widetilde{M}) : \tau^*\alpha = \alpha \right\}$$

is a bijection.

Note: to convince ourselves that $\pi_*\alpha$ is smooth, we can look in local coordinates $(\tilde{U}, \tilde{x})$ on $\widetilde{M}$ where the local diffeomorphism π takes the form

$$\pi(\tilde{x}^1, \dots, \tilde{x}^n) \stackrel{\text{loc}}{=} (\tilde{x}^1, \dots, \tilde{x}^n).$$

Then if

$$\alpha \stackrel{\text{loc}}{=} \frac{1}{k!} \alpha_{i_1 \dots i_k}(\tilde{x}) d\tilde{x}^{i_1} \wedge \dots \wedge d\tilde{x}^{i_k}$$

then

$$\pi_*\alpha \stackrel{\text{loc}}{=} \frac{1}{k!} \alpha_{i_1 \dots i_k}(\tilde{x}) d\tilde{x}^{i_1} \wedge \dots \wedge d\tilde{x}^{i_k}.$$

□