

Problem Set 6: Metrics and Integration

Problem 1. Let (M, g) be a manifold with metric. The gradient of a function ∇f is a vector field defined by $(df)^\sharp$, or in local coordinates

$$\nabla f \stackrel{\text{loc}}{=} \left[g^{ik} \partial_k f \right] \frac{\partial}{\partial x^i}.$$

Compute ∇f in polar coordinates over $\mathbb{R}^2 \setminus \{0\}$ with the Euclidean metric.

Problem 2. In this problem, we rederive formulas from calculus on arclength and surface area.

(a) Let $f : (a, b) \rightarrow \mathbb{R}$ and consider the submanifold

$$S = \{(x, f(x)) : x \in (a, b)\} \subset \mathbb{R}^2.$$

In other words, S is the graph of $y = f(x)$. Equip S with coordinates $\varphi : S \rightarrow U$, $U = (a, b)$ given by

$$\varphi(x, f(x)) = x.$$

Denote by g the Euclidean metric on $\mathbb{R}^2$ restricted to S . Find the expression for g in the coordinate chart (U, x) . To be clear, this means $g = (\varphi^{-1})^* g_{\text{Euc}}$. Derive the formula for the arclength by simplifying the general formula

$$\int_U \sqrt{\det g_{ij}} dx^1 \cdots dx^n$$

in this context.

(b) Next, let $f : U \rightarrow \mathbb{R}$ be a function of two variables $f(x, y)$, where $U \subset \mathbb{R}^2$ is an open set. Consider the submanifold

$$S = \{(x, y, f(x, y)) : (x, y) \in U\} \subset \mathbb{R}^3.$$

Equip S with coordinates $\varphi : S \rightarrow U$. Compute the metric g in these coordinates, where g is the pullback of the Euclidean metric. Derive the formula for the area of a surface by simplifying the general formula

$$\int_U \sqrt{\det g_{ij}} dx^1 \cdots dx^n$$

in this context.

(c) Suppose the level set of the function $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ defines a submanifold:

$$S = \{F(x, y, z) = 0\} \subseteq \mathbb{R}^3.$$

Let $(a, b, c) \in S$ and suppose $\partial_z F(a, b, c) \neq 0$. By the implicit function theorem, there exists a neighborhood $U \subset \mathbb{R}^2$ of (a, b) , $V \subset \mathbb{R}^3$ of (a, b, c) and a function $f : U \rightarrow \mathbb{R}$ such that

$$S \cap V = \{(x, y, f(x, y)) : (x, y) \in U\}.$$

Use the results from part (b) to derive the formula for area of $S \cap V$:

$$\iint_U \frac{|\nabla F|}{|\partial_z F|} dx dy.$$

Verify this formula by computing the area of the hemisphere by taking $F = x^2 + y^2 + z^2 - 1$ and $U = \{x^2 + y^2 < 1\}$.

Problem 3. Let $0 < b < a$. Let $f : [0, 2\pi] \times [0, 2\pi] \rightarrow \mathbb{R}^3$ be given by

$$f(u, v) = ((a + b \cos u) \cos v, (a + b \cos u) \sin v, b \sin u).$$

This is a parametrization of the torus of revolution T^2 .

(a) Compute the metric g_{ij} on T^2 in (u, v) -coordinates, where the metric is induced by the Euclidean metric on $\mathbb{R}^3$. Give your answer as a 2×2 matrix function of (u, v) .

(b) Compute the length of the curves $c_1 : [0, 2\pi] \rightarrow T^2$ given by

$$c_1(t) = f(\pi, t),$$

and $c_2 : [0, 2\pi] \rightarrow T^2$ given by $c_2(t) = f(t, \pi)$.

(c) Compute the area of the torus.

Problem 4. Show that the manifold TM is always orientable, even if M is not.

Problem 5. Let $L \rightarrow \mathbb{RP}^1$ be the tautological line bundle

$$L = \left\{ ([x_0, x_1], w) \in \mathbb{RP}^1 \times \mathbb{R}^2 : w \in [x_0, x_1] \right\}.$$

Recall that the transition function is $\tau_{01} = x_0/x_1$. Show that the manifold L is not orientable.

Problem 6. Let $T^2 = S^1 \times S^1 \subseteq \mathbb{R}^4$ be given by

$$T^2 = \left\{ (w, x, y, z) \in \mathbb{R}^4 : w^2 + x^2 = y^2 + z^2 = 1 \right\}.$$

Let

$$\omega = xz \, dw \wedge dy, \quad \omega \in \Omega^2(\mathbb{R}^4).$$

Compute the integral

$$\int_{T^2} \omega.$$

You should declare an orientation on T^2 before evaluating this integral.

Problem 7. Let $\tilde{\omega} \in \Omega^2(\mathbb{R}^3)$ be defined by

$$\tilde{\omega} = x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy.$$

Let $\iota : S^2 \rightarrow \mathbb{R}^3$ and $\omega = \iota^* \tilde{\omega} \in \Omega^2(S^2)$. Give S^2 the metric $g = \iota^* g_{\text{Euc}}$ and the outward-pointing orientation via $S^2 = \partial B^3$.

(a) Show that $\omega = d\text{vol}_g$.

(b) Compute the integral $\int_{S^2} \omega$.

(c) Show that there does not exist $\alpha \in \Omega^1(S^2)$ such that $\omega = d\alpha$.

Problem 8. Let $M = S^2 \cap \{z \geq 0\}$ be the upper hemisphere with outward orientation and inclusion $\iota : M \hookrightarrow \mathbb{R}^3$. Let

$$\tilde{\alpha} = 2dx \wedge dy - dx \wedge dz, \quad \tilde{\alpha} \in \Omega^2(\mathbb{R}^3),$$

and define $\alpha = \iota^* \tilde{\alpha}$. Compute the surface integral $\int_M \alpha$.

Problem 9. Cover S^1 by the two charts (U, θ) and $(\tilde{U}, \tilde{\theta})$ given by

$$U = \left\{ e^{i\theta} : \theta \in (0, 2\pi) \right\}, \quad \tilde{U} = \left\{ e^{i\tilde{\theta}} : \tilde{\theta} \in (-\pi, \pi) \right\}.$$

Let $c \in \mathbb{R}$ and consider the 1-forms

$$\alpha = \sin^2 \theta \, d\theta, \quad \beta = c \, d\theta.$$

Find c and $f \in C^\infty(S^1)$ such that $\alpha - \beta = df$.

Problem 10. Consider the 1-form

$$\alpha = -\frac{y \, dx}{x^2 + y^2} + \frac{x \, dy}{x^2 + y^2}.$$

defined on $\mathbb{R}^2 \setminus \{0\}$.

(a) Show that $d\alpha = 0$.

(b) Show that there does not exist $f : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$ such that $\alpha = df$.

(c) Find an explicit $f : \mathbb{R}^2 \setminus \{x = 0\} \rightarrow \mathbb{R}$ such that $\alpha = df$. Hint: consider $\arctan u$.
